EMPG–15–06

Triproducts, nonassociative star products and geometry of *R*-flux string compactifications*

Paolo Aschieri[1] and **Richard J. Szabo**[2]

[1] *Dipartimento di Scienze e Innovazione Tecnologica*
and *INFN Torino, Gruppo collegato di Alessandria*
Università del Piemonte Orientale
Viale T. Michel 11, 15121 Alessandria, Italy
Email: `aschieri@unipmn.it`

[2] *Department of Mathematics, Heriot-Watt University*
Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, U.K.
and *Maxwell Institute for Mathematical Sciences, Edinburgh, U.K.*
and *The Higgs Centre for Theoretical Physics, Edinburgh, U.K.*
Email: `R.J.Szabo@hw.ac.uk`

Abstract

We elucidate relations between different approaches to describing the nonassociative deformations of geometry that arise in non-geometric string theory. We demonstrate how to derive configuration space triproducts exactly from nonassociative phase space star products and extend the relationship in various directions. By foliating phase space with leaves of constant momentum we obtain families of Moyal–Weyl type deformations of triproducts, and we generalize them to new triproducts of differential forms and of tensor fields. We prove that nonassociativity disappears on-shell in all instances. We also extend our considerations to the differential geometry of nonassociative phase space, and study the induced deformations of configuration space diffeomorphisms. We further develop general prescriptions for deforming configuration space geometry from the nonassociative geometry of phase space, thus paving the way to a nonassociative theory of gravity in non-geometric flux compactifications of string theory.

*Contribution to the Proceedings of "Conceptual and Technical Challenges for Quantum Gravity", September 8–12, 2014, Sapienza University of Rome, Italy; to be published in *Journal of Physics Conference Series*, eds. G. Amelino-Camelia, P. Martinetti and J.-C. Wallet.

Contents

1 Introduction and summary

Noncommutative geometry has long been believed to provide a framework for understanding the generalizations of classical spacetime geometry required to describe Planck scale quantum geometry, and ultimately quantum gravity. Yet its precise relation with other approaches to quantum gravity such as string theory have remained somewhat elusive. The surge of interest in noncommutative geometry from string theory originally came about in the *open* string sector, where the massless bosonic modes contain gauge and scalar fields: It was observed that D-brane worldvolumes acquire a noncommutative deformation in the background of a non-zero B-field [13, 31, 32], which is moreover nonassociative when the flux $H = \mathrm{d}B$ is non-vanishing [15]; the low-energy worldvolume effective field theory is then described by a noncommutative gauge theory. However, this realization does not really shed light on connections with the *closed* string sector whose massless bosonic modes contain gravitational degrees of freedom such as the metric g, the B-field, and the dilaton ϕ. Vanishing of the beta-functions for these fields (at one-loop) as required by conformal invariance of the worldsheet field theory yields equations of motion which can be derived from the standard bosonic closed string low-energy effective action

$$S = \int_M R(g) - \frac{1}{12}\, \mathrm{e}^{-\phi/3}\, H \wedge *_g\, H - \frac{1}{6}\, \mathrm{d}\phi \wedge *_g\, \mathrm{d}\phi \ . \tag{1.1}$$

It can be hoped that an equivalent noncommutative/nonassociative version of this effective field theory provides a suitable target space framework in which to address problems related to quantum gravity.

A precise connection between noncommutative geometry and the closed string sector has been found recently through non-geometric flux compactifications [21, 33]. The prototypical

example involves a three-torus with constant three-form H-flux; T-dualizing along its cycles gives rise to constant geometric and non-geometric fluxes which can be depicted schematically through the T-duality chain

$$H_{abc} \xrightarrow{T_a} f^a{}_{bc} \xrightarrow{T_b} Q^{ab}{}_c \xrightarrow{T_c} R^{abc} . \tag{1.2}$$

While the H-flux and metric f-flux backgrounds can be described globally as Riemannian manifolds, the Q-flux background involves T-duality transformations as transition functions between local trivializations of its tangent bundle and gives rise to a T-fold, while the R-flux background is a purely non-geometric string vacuum as its metric and B-field are not even locally defined because they depend on the winding coordinates of the dual space. These non-geometric flux compactifications have been recently purported to have a global description in terms of noncommutative and nonassociative structures [24, 10, 12, 26] (see [28] for a recent review and further references).

Just as in the case of open strings, there are two ways in which one can see the appearence of nonassociativity of target space coordinates in the R-flux background: Either through canonical analysis of the closed string coordinates or by careful examination of worldsheet scattering amplitudes. In the formulation of [24], nonassociativity of the coordinates of *configuration space* arises as a failure of the Jacobi identity of the bracket of canonical variables on *phase space*. This model is described in Section 2, and it defines a twisted Poisson structure on phase space which can be quantized using Kontsevich formality to obtain an explicit nonassociative star product of functions for deformation quantization of phase space [26]. The nonassociative star product on phase space defines nonassociative tori in closed string theory. If we regard it as a substitute for canonical quantization as suggested by [6], then it may be used to formulate a consistent nonassociative version of quantum mechanics [27]. On the other hand, from a worldsheet perspective an exact analysis of conformal field theory correlation functions can be carried out to linear order in the H-flux, wherein the backreaction due to the curvature of spacetime can be neglected. This was performed by [12] where off-shell traces of nonassociativity were found and packaged into "triproducts" of fields directly on configuration space at linear order in R; at this order the elementary triproduct of three fields was shown by [26] to be reproduced by the associator of the phase space star product. In [12] an all orders exponential formula for arbitrary n-triproducts involving n fields was moreover conjectured, however its verification through a conformal field theory analysis on flat space is not possible.

In this contribution we fill a technical gap in these two approaches to the nonassociative geometry of R-flux compactifications by explicitly *deriving* the conjectural all orders configuration space n-triproducts of [12] from the phase space star products of [26]. As their origins are quite different, this provides a non-trivial confirmation of these triproducts and moreover of the validity of the phase space formulation as the fundamental theory of closed strings propagating in the non-geometric R-flux compactification, which is the perspective advocated by [26] and pursued in the present paper wherein phase space appears as the effective closed string target space. Physical applications of this perspective to scalar field theories can be found in [25].

For later reference, let us briefly review the phase space approach. The starting point of [26] is the Courant sigma-model based on the standard Courant algebroid $C = TM \oplus T^*M$ over the target space M with natural frame $(\psi_I) = (\partial_i, \mathrm{d}x^i)$; the structure maps consist of the fibre

metric h_{IJ} with $h_{ij} = \langle \partial_i, \mathrm{d}x^j \rangle = \delta_i{}^j$, the anchor matrix $\rho(\psi_I) = P_I{}^i(x)\,\partial_i$,[1] and the three-form $T_{IJK}(x) = [\psi_I, \psi_J, \psi_K]_C$. This data can be used to construct a topological field theory on a three-dimensional M2-brane worldvolume Σ_3 with action given by

$$S_T = \int_{\Sigma_3} \phi_i \wedge \mathrm{d}X^i + \frac{1}{2}\,h_{IJ}\,\alpha^I \wedge \mathrm{d}\alpha^J - P_I{}^i(X)\,\phi_i \wedge \alpha^I + \frac{1}{6}\,T_{IJK}(X)\,\alpha^I \wedge \alpha^J \wedge \alpha^K \ , \tag{1.3}$$

where $X : \Sigma_3 \to M$ is the M2-brane embedding, and $\alpha \in \Omega^1(\Sigma_3, X^*C)$ and $\phi \in \Omega^2(\Sigma_3, X^*T^*M)$ are auxilliary fields. By taking T to be any of the fluxes in (1.2), one can look at the effective dynamics of the membrane fields; putting in all of the fluxes simultaneously and independently of one another (e.g., by replacing M with its double manifold as in double field theory) could then give a precise formulation of the formal T-duality chain (1.2) in terms of (quantum) gauge symmetries of the Courant sigma-model. In the case of a purely geometric H-flux background it is shown in [26] that the M2-brane model reduces to the standard H-twisted Poisson sigma-model with target space M on the boundary $\Sigma_2 = \partial\Sigma_3$, which describes closed string propagation in the configuration space M with a non-constant B-field. On the other hand, in the case of a non-geometric constant R-flux the M2-brane model reduces instead to a generalized Poisson sigma-model on Σ_2 whose action can be expressed in linearized form by using auxilliary one-form fields η_I as

$$S_R = \int_{\Sigma_2} \eta_I \wedge \mathrm{d}X^I + \frac{1}{2}\,\Theta^{IJ}(X)\,\eta_I \wedge \eta_J \ , \tag{1.4}$$

where $X = (X^i, P_i) : \Sigma_2 \to T^*M$ now embed closed strings into an effective target space which coincides with the phase space of M, and

$$\Theta = \begin{pmatrix} R^{ijk}\,p_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \tag{1.5}$$

is a twisted Poisson bivector which describes a noncommutative and nonassociative phase space (see Section 2). This point of view that closed strings should propagate in phase space rather than configuration space is also pursued in a more general context by [17] wherein it is argued that the fundamental symmetry of string theory contains phase space diffeomorphisms in order to accomodate T-duality and other stringy features.

In Section 3 we shall in fact derive a generalization to families of Moyal–Weyl type deformations of the triproducts which are induced from a foliation of phase space by leaves of constant momentum and describe their physical significance; the leaf of zero momentum yields the triproducts of [12]. We further give an explicit proof that the integrated versions of these deformed triproducts coincide with those of the associative Moyal–Weyl star products, thus reproducing the physical expectations that nonassociativity is not visible in on-shell conformal field theory correlation functions. This is a remarkable cancellation that happens due to specific orderings of bracketing phase space star products. In Section 4 we generalize these results to new (deformed) triproducts of arbitrary differential forms via nonassociative deformations of phase space exterior products following the cochain twist deformation formalism proposed by [27] for the study of non-geometric fluxes. These algebraic techniques, originally developed to study quasi-Hopf algebra symmetries [16], are further extended to study the differential geometry of nonassociative phase space, i.e., the deformed Lie algebra of infinitesimal diffeomorphisms and

[1] Implicit summation over repeated upper and lower indices is understood throughout.

their Lie derivative action on exterior forms and tensor fields. This geometry is then induced on each constant momentum leaf leading in particular to a deformed Lie algebra of infinitesimal diffeomorphisms on configuration space.

While the results of the present paper are mostly technical, they offer further insight into the nature of nonassociative geometry in non-geometric string theory in at least three ways. Firstly, we give a fairly general prescription for inducing configuration space triproducts from nonassociative star products on phase space and clarify in what sense these products obey the physical expectations of cyclicity of on-shell string scattering amplitudes, though a complete geometric framework for choosing appropriate section slices remains to be worked out. The different leaves correspond to a multitude of choices of Bopp shifts in phase space and are reminescent of the section conditions in double field theory (the weak and strong constraints). Secondly, our considerations based on cochain twist theory are general enough that they may be applicable beyond the flat space limit to the *curved* space triproducts conjectured by [11] for non-constant fluxes, and hence they may provide a precise link between the phase space formalism and the framework of double field theory; such a connection is explored by [17] in a related but more general context. Thirdly, our derivation and considerations of new triproducts involving arbitrary differential forms and infinitesimal diffeomorphisms on configuration space is a first step to understanding the construction of nonassociative deformations of gravity and their physical relevance in on-shell string theory.

2 Nonassociative geometry of R-space

2.1 Phase space formulation

We shall begin by describing the phase space model for the non-geometric R-flux background and discuss some ways in which it may be interpreted as a deformation of the geometry of configuration space. Let $M = \mathbb{R}^d$ be the decompactification limit of a d-torus, endowed with a constant three-form H-flux. As reviewed in Section 1, via T-duality this space is mapped to a non-geometric string background with a constant trivector R-flux $R = \frac{1}{3!} R^{ijk}\, \partial_i \wedge \partial_j \wedge \partial_k$, where $\partial_i = \frac{\partial}{\partial x^i}$ in local coordinates $x = (x^i) \in \mathbb{R}^d$. Explicit string and conformal field theory computations show that the string geometry acquires a noncommutative and nonassociative deformation for closed strings which wind and propagate in the non-geometric background [10, 24, 12, 14, 2]. An explicit realization of such a nonassociative deformation of the spacetime geometry is provided by the phase space description of the parabolic R-flux model on the cotangent bundle $\mathcal{M} = T^*M = M \times (\mathbb{R}^d)^*$ with local coordinates (x, p), where $p = (p_i) \in (\mathbb{R}^d)^*$. In this setting, the deformation is described by a commutator algebra for the local phase space coordinates given by

$$[x^i, x^j] = \frac{\mathrm{i}\,\ell_s^4}{3\hbar}\, R^{ijk}\, p_k \ , \qquad [x^i, p_j] = \mathrm{i}\,\hbar\, \delta^i{}_j \qquad \text{and} \qquad [p_i, p_j] = 0 \ , \tag{2.1}$$

which has a non-trivial Jacobiator

$$[x^i, x^j, x^k] = \ell_s^4\, R^{ijk} \ . \tag{2.2}$$

In the point particle limit $\ell_s = 0$ this is just the usual Lie algebra of ordinary quantum phase space.

As anticipated by [10, 24, 11], and proven in [27] directly from the phase space model, the R-flux background does not permit the notion of a point due to a minimal volume which enters an uncertainty relation for the position coordinates given by

$$\Delta x^i \, \Delta x^j \, \Delta x^k \geq \tfrac{1}{2} \, \ell_s^4 \, R^{ijk} \ . \tag{2.3}$$

This lack of a notion of a point illustrates why the R-flux compactification is not even locally geometric; this is evident in the phase space model which, as a result of T-duality, requires both position coordinates and momenta in any local description. Hence it is not clear how to formulate a gravity theory, or any other field theory, on this nonassociative space. An approach based on fundamental loop space variables, rather than functions on M, was pursued in [29, 30]; the usage of such variables is also mentioned in [11]. This approach reflects the "non-locality" of the non-geometric R-space. It is natural in the lift of Type IIA string theory to M-theory in which the (closed string) boundary of an open M2-brane ending on an M5-brane in a constant C-field background gives rise to a noncommutative loop space algebra on the M5-brane worldvolume [8, 23], which corresponds to the noncommutativity and nonassociativity felt by a fundamental closed string in a constant H-flux; this perspective is utilized in the formulation by [26] of closed string propagation in the non-geometric R-flux background using M2-brane degrees of freedom, as reviewed in Section 1, and it connects open and closed string noncommutative geometry.

Another set of fundamental variables is obtained by considering the algebra $\mathsf{Diff}(M)$ of (formal) differential operators on M with typical elements of the form

$$\underline{f}(x) = f(x) + \sum_{k=1}^{\infty} f^{i_1 \cdots i_k}(x) \, \partial_{i_1} \cdots \partial_{i_k} \ , \tag{2.4}$$

where $f, f^{i_1 \cdots i_k} \in A := C^\infty(M)$ and $\underline{f}(x) g(x) = f(x) \, g(x) + \sum_k f^{i_1 \cdots i_k}(x) \, \partial_{i_1} \cdots \partial_{i_k} g(x)$ for all $g \in A$. From the last two commutation relations in (2.1) we can identify $p_i = -\,\mathrm{i}\,\hbar\,\partial_i$ and the nonassociativity relations as relations involving differential operators

$$[x^i, x^j] = \tfrac{1}{3} \, \ell_s^4 \, R^{ijk} \, \partial_k \ . \tag{2.5}$$

Such an interpretation was advocated by [19, 18] in the (associative) context of open strings on D-branes in non-constant B-field backgrounds.

A related interpretation of the twisted Poisson structure (2.1) on phase space $\mathcal{M} = T^*M$, as discussed in [26, Section 2.5], is that of a higher Poisson structure on the multivector field algebra $\mathsf{Vect}^\bullet(M) = C^\infty(M, \bigwedge^\bullet TM)$.

2.2 Deformations of configuration space geometry

In order to formulate a nonassociative theory of gravity on the original configuration space M, we note that there is a natural foliation of $\mathcal{M}$ over M defined by the (global) sections $s_{\bar p} : M \to \mathcal{M}$ with $s_{\bar p}(x) = (x, \bar p)$ for $\bar p \in (\mathbb{R}^d)^*$; this simply means that we can realise M as any leaf of constant momentum in phase space. We can therefore restrict functions on phase space $\underline{f} \in C^\infty(\mathcal{M})$ to $C^\infty(M)$ via the family of pullbacks $f_{\bar p} := s_{\bar p}^* \underline{f}$. Thus via pullback along the bundle projection $\pi : \mathcal{M} \to M$ we can transport functions on M into the cotangent bundle, perform the necessary nonassociative deformations on $\mathcal{M}$, and then pullback along $s_{\bar p}$ to obtain

the desired nonassociative deformations of the geometry of M; schematically, we can depict this foliation by quantizations of the configuration manifold M via the commutative diagrams

$$
\begin{tikzcd}
C^\infty(\mathcal{M}) \arrow[r, "\mathscr{Q}"] & \widehat{C^\infty(\mathcal{M})} \arrow[d, "s^*_{\bar p}"] \\
C^\infty(M) \arrow[u, "\pi^*"] \arrow[r, dotted, "\mathscr{Q}_{\bar p}"'] & \widehat{C^\infty(M)}
\end{tikzcd}
\tag{2.6}
$$

with $s^*_{\bar p}\circ\pi^* = (\pi\circ s_{\bar p})^* = \mathrm{id}$, where $\mathscr{Q}$ is the nonassociative quantization of $\mathcal{M}$ whose foundations are developed by [26, 27], while $\mathscr{Q}_{\bar p}$ is the desired nonassociative quantization of M induced by the diagram. We can use this perspective to obtain nonassociative field theories on M via this systematic implementation of the nonassociative geometry of $\mathcal{M}$.

This perspective also suggests a way in which to obtain richer deformations of the geometry of configuration space from the nonassociative quantization of phase space. By pullbacks along π^* we consider fields $\underline{f}\in C^\infty(\mathcal{M})$ that satisfy the *section constraints*

$$
\tilde\partial^i\,\underline{f} = 0 \qquad \text{for} \quad i=1,\dots,d\ , \tag{2.7}
$$

where $\tilde\partial^i := \frac{\partial}{\partial p_i}$. There is a natural invariant metric on phase space $\mathcal{M}=T^*M$ which in a local frame is defined by

$$
\gamma = \mathrm{d}x^i\otimes \mathrm{d}p_i + \mathrm{d}p_i\otimes \mathrm{d}x^i\ . \tag{2.8}
$$

It defines an $O(d,d)$-structure on $\mathcal{M}$, i.e., a reduction of the structure group $GL(2d,\mathbb{R})$ of the tangent bundle $T\mathcal{M}$ to the subgroup $O(d,d)$; this is the symmetry group underlying quantum mechanical Born reciprocity and its relation to T-duality is explained by [17]. Drawing now on the evident similarities with double field theory (see [1, 20] for reviews and [34] for a rigorous mathematical treatment), a natural weakening of the section condition (2.7) involves constraints on phase space functions $\underline{f}$ which remove their momentum dependence locally only up to an $O(d,d)$ transformation. For example, if θ is any constant nondegenerate bivector on M, then this constraint is solved by any phase space function $\underline{f}$ which depends only on the non-local Bopp shifts $x+\theta\cdot p$; this d-dimensional slice of the $2d$-dimensional phase space geometry can be rotated to the constant momentum leaves above via an $O(d,d)$ transformation. In particular, the tangent bundle on phase space decomposes as $T\mathcal{M}\cong L\oplus L^*$ where L is the tangent bundle on the leaves of the foliation and L^* is its dual bundle with respect to the orthogonal complement in the metric (2.8). The section constraints on functions $\underline{f}\in C^\infty(\mathcal{M})$ and more generally on covariant tensor fields $\underline{T}\in C^\infty(\mathcal{M},\bigotimes^\bullet T^*\mathcal{M})$ now read as

$$
Z(\,\underline{f}\,) = 0 \qquad \text{and} \qquad \imath_Z\,\underline{T} = 0 = \mathcal{L}_Z\,\underline{T} \tag{2.9}
$$

for all sections $Z\in C^\infty(\mathcal{M},L^*)$, where $\imath$ denotes contraction and $\mathcal{L}$ is the Lie derivative. This means that the set of admissible fields is constrained to *foliated* tensor fields with respect to the distribution L^*. For the foliation defined by $s_{\bar p}$, one has $L=TM$ and the section constraints can be imposed by taking $Z=\tilde\partial^i$ for each $i=1,\dots,d$. A similar perspective on closed string target spaces is addressed in [17] where connections with double field theory are also explored.

These more general foliations of phase space could lead to much richer classes of deformations of M, because the nonassociative quantizations $\mathscr{Q}$ are *not* $O(d,d)$-invariant, as is evident from

the defining relations (2.1). Moreover, quantization of the section conditions (2.9) themselves could lead to interesting deformations of the foliated tensor fields on the d-dimensional slices. It would be interesting to repeat the analysis of this paper for such more general section slices. Among other things, this could help elucidate possible relationships between the noncommutative gerbe structure [3] on phase space underlying the nonassociative deformations [26] and the abelian gerbe structures underlying the generalised manifolds in double geometry [9, 22]. It should also help in understanding how to lift geometric objects from M to its cotangent bundle $\mathcal{M} = T^*M$ in a way suited to describe nonassociative deformations of the geometry and of gravity directly on the configuration manifold M.

3 Triproducts from phase space star products

3.1 Families of n-triproducts

We shall now derive triproduct formulas in the context of Section 2 which include those of [12] as special cases. Our starting point is the nonassociative star product $\star$ on phase space derived by [26] (see also [6]) which quantizes the commutation relations (2.1). Here we shall utilize the expression for this star product as a twisted convolution product (see Section 4.3 for a different derivation). For phase space functions $\underline{f}, \underline{g} \in C^\infty(\mathcal{M})$, the integral formula derived in [27, eq. (3.38)] adapted to the normalizations in (2.1) reads as

$$\begin{aligned}(\underline{f} \star \underline{g})(x,p) &= \frac{1}{(\pi\,\hbar)^{2d}} \iint_M \mathrm{d}^d z\ \mathrm{d}^d z' \iint_{M^*} \mathrm{d}^d k\ \mathrm{d}^d k'\ \underline{f}(x+z,p+k)\, \underline{g}(x+z',p+k') \\ &\qquad \times\ \mathrm{e}^{-\frac{2\,\mathrm{i}}{\hbar}\,(k\cdot z'-k'\cdot z)}\ \mathrm{e}^{-\frac{2\,\mathrm{i}\,\ell_s^4}{3\hbar^3}\,R(k,k',p)}\ , \qquad (3.1)\end{aligned}$$

where $k'\cdot z := k'_i\, z^i$ and $R(k,k',p) := R^{ijl}\, k_i\, k'_j\, p_l$. Restricted to functions $f,g \in C^\infty(M)$ by the pullback along $\pi : \mathcal{M} \to M$, after rescaling, this product reads as

$$\begin{aligned}(\pi^* f \star \pi^* g)(x,p) &= \frac{1}{(2\pi)^{2d}} \iint_M \mathrm{d}^d z\ \mathrm{d}^d z'\ f(x+z)\, g(x+z') \\ &\qquad \times \iint_{M^*} \mathrm{d}^d k\ \mathrm{d}^d k'\ \mathrm{e}^{-\mathrm{i}\,(k\cdot z'-k'\cdot z)}\ \mathrm{e}^{-\frac{\mathrm{i}\,\ell_s^4}{6\hbar}\,R(k,k',p)}\ . \qquad (3.2)\end{aligned}$$

In general, the star product of two fields on M is a field on $\mathcal{M}$, i.e., a differential operator on M. Thus the star product $\star$ does not close in the algebra $A = C^\infty(M)$. We can define a family of products of fields on M by pulling this product back under the local sections $s_{\bar p} : M \to \mathcal{M}$ that foliate $\mathcal{M}$ with leaves isomorphic to M. This defines a family of 2-products $\mu^{(2)}_{\bar p} : A \otimes A \to A$ given by

$$\begin{aligned}\mu^{(2)}_{\bar p}(f \otimes g)(x) &:= s^*_{\bar p}\,(\pi^* f \star \pi^* g)(x,p) \\ &= \frac{1}{(2\pi)^{2d}} \iint_M \mathrm{d}^d z\ \mathrm{d}^d z'\ f(x+z)\, g(x+z') \\ &\qquad \times \iint_{M^*} \mathrm{d}^d k\ \mathrm{d}^d k'\ \mathrm{e}^{-\mathrm{i}\,(k\cdot z'-k'\cdot z)}\ \mathrm{e}^{-\frac{\mathrm{i}\,\ell_s^4}{6\hbar}\,\theta_{\bar p}(k,k')} \\ &= \frac{1}{(2\pi)^{2d}} \iint_{M^*} \mathrm{d}^d k\ \mathrm{d}^d k'\ \mathrm{e}^{\,\mathrm{i}\,(k+k')\cdot x}\ \mathrm{e}^{\frac{\mathrm{i}\,\ell_s^4}{6\hbar}\,\theta_{\bar p}(k,k')}\end{aligned}$$

$$
\times \iint_M \mathrm{d}^d z \ \mathrm{d}^d z' \ f(z) \, g(z') \ \mathrm{e}^{-\mathrm{i}\,(k\cdot z' + k'\cdot z)}
$$

$$
= \int_{M^*} \mathrm{d}^d k \ \int_{M^*} \mathrm{d}^d k' \ \hat f(k') \, \hat g(k) \, \mathrm{e}^{\,\mathrm{i}\,(k+k')\cdot x} \ \mathrm{e}^{\,\frac{\mathrm{i}\,\ell_s^4}{6\hbar}\,\theta_{\bar p}(k,k')}
$$

$$
= \int_{M^*} \mathrm{d}^d k \ \int_{M^*} \mathrm{d}^d k' \ \hat f(k) \, \hat g(k') \, \mathrm{e}^{\,\mathrm{i}\,(k+k')\cdot x} \ \mathrm{e}^{\,-\frac{\mathrm{i}\,\ell_s^4}{6\hbar}\,\theta_{\bar p}(k,k')} \ , \tag{3.3}
$$

where

$$
\theta_{\bar p} = \tfrac12\, \theta^{ij}_{\bar p} \, \partial_i \wedge \partial_j := \tfrac12\, R^{ijk}\, \bar p_k \, \partial_i \wedge \partial_j \tag{3.4}
$$

are constant bivectors on M along the leaves of constant momentum in phase space, and

$$
\hat f(k) = \frac{1}{(2\pi)^d} \int_M \mathrm{d}^d x \ f(x) \ \mathrm{e}^{-\mathrm{i}\,k\cdot x} \tag{3.5}
$$

is the Fourier transform of the function $f \in C^\infty(M)$. This last expression is simply the momentum space representation of the associative Moyal–Weyl star product $\star_{\bar p}$ determined by the bivector $-\frac{\ell_s^4}{3\hbar}\,\theta_{\bar p}$, and we thus find an expression for the 2-products in terms of a bidifferential operator

$$
\mu^{(2)}_{\bar p}(f\otimes g) = f \star_{\bar p} g := \mu\Big(\exp\big(\tfrac{\mathrm{i}\,\ell_s^4}{6\hbar}\, \theta^{ij}_{\bar p}\, \partial_i \otimes \partial_j\big)(f\otimes g)\Big) \tag{3.6}
$$

where $\mu(f\otimes g) = f\,g$ is the pointwise multiplication of functions in $A = C^\infty(M)$.

We shall now generalize this result to n-products $\mu^{(n)}_{\bar p} : A^{\otimes n} \to A$ of functions $f_1,\dots,f_n \in A$ for $n \geq 3$, which we define in an analogous way. However, we must keep in mind that the phase space star product $\star$ is nonassociative, so we have to keep track of the order in which we group binary products of functions on $\mathcal{M}$; this is even true after integration over $\mathcal{M}$ [27]. We choose the ordering

$$
\mu^{(n)}_{\bar p}(f_1\otimes\cdots\otimes f_n) := s^*_{\bar p}\big[\big((\cdots(\pi^* f_1 \star \pi^* f_2)\star \pi^* f_3)\star\cdots\big)\star \pi^* f_n\big] \ , \tag{3.7}
$$

and discuss the significance of the other orderings below. We shall compare this n-product to iterations of the Moyal–Weyl star products given by

$$
\star_{\bar p}(f_1\otimes\cdots\otimes f_n) := f_1 \star_{\bar p} \cdots \star_{\bar p} f_n = \mu\Big[\exp\Big(\frac{\mathrm{i}\,\ell_s^4}{6\hbar} \sum_{1\leq a<b\leq n} \theta^{ij}_{\bar p}\, \partial_i^a\, \partial_j^b\Big)(f_1\otimes\cdots\otimes f_n)\Big] \ , \tag{3.8}
$$

where $\partial_i^a = (\mathrm{id}\otimes\cdots\otimes\partial_i\otimes\cdots\otimes\mathrm{id})$, $a=1,\dots,n$ denotes the derivative ∂_i acting on the a-th factor of the tensor product $f_1\otimes\cdots\otimes f_n$; note that no bracketings need be specified here due to associativity of the products $\star_{\bar p}$ for all $\bar p$.

Proposition 3.9.

$$
\mu^{(n)}_{\bar p}(f_1\otimes\cdots\otimes f_n) = \star_{\bar p}\Big[\exp\Big(\frac{\ell_s^4}{12} \sum_{1\leq a<b<c\leq n} R^{ijk}\, \partial_i^a\, \partial_j^b\, \partial_k^c\Big)(f_1\otimes\cdots\otimes f_n)\Big] \ .
$$

Proof. By iterating the integral formula (3.1), it is straightforward to derive the integral representation

$$
s^*_{\bar p}\big[\big((\cdots(\pi^* f_1 \star \pi^* f_2)\star \pi^* f_3)\star\cdots\big)\star \pi^* f_n\big](x)
$$

$$
\begin{aligned}
&= \frac{1}{(\pi\,\hbar)^{2(n-1)d}} \prod_{a=1}^{n-1} \iint_M \mathrm{d}^d z_a\ \mathrm{d}^d z'_a \iint_{M^*} \mathrm{d}^d k_a\ \mathrm{d}^d k'_a\ \mathrm{e}^{-\frac{2\,\mathrm{i}}{\hbar}\,(k_a\cdot z'_a - k'_a\cdot z_a)} \\
&\qquad \times \prod_{b=1}^{n} f_b(x+z_1+\cdots+z_{n-b}+z'_{n-b+1}) \\
&\qquad\qquad \times\ \exp\Big(-\frac{2\,\mathrm{i}\,\ell_s^4}{3\hbar^3} \sum_{c=1}^{n-1} R(k_c,k'_c,p+k_1+\cdots+k_{c-1})\Big)
\end{aligned} \tag{3.10}
$$

with the conventions $z_0=0, z'_n=0, k_0=0$. After rescaling we thus get

$$
\begin{aligned}
\mu_{\bar p}^{(n)}(f_1\otimes\cdots\otimes f_n)(x) &= \frac{1}{(2\pi)^{2(n-1)d}} \prod_{a=1}^{n-1} \iint_M \mathrm{d}^d z_a\ \mathrm{d}^d z'_a \iint_{M^*} \mathrm{d}^d k_a\ \mathrm{d}^d k'_a\ \mathrm{e}^{-\,\mathrm{i}\,(k_a\cdot z'_a - k'_a\cdot z_a)} \\
&\qquad \times \prod_{b=1}^{n} f_b(x+z_1+\cdots z_{n-b}+z'_{n-b+1}) \\
&\qquad\qquad \times\ \exp\Big(-\frac{\mathrm{i}\,\ell_s^4}{12} \sum_{c=1}^{n-1} R(k_c,k'_c,k_1+\cdots+k_{c-1})\Big) \\
&\qquad\qquad\qquad \times\ \exp\Big(-\frac{\mathrm{i}\,\ell_s^4}{6\hbar} \sum_{c=1}^{n-1} \theta_{\bar p}(k_c,k'_c\,)\Big) \\
&= \frac{1}{(2\pi)^{2(n-1)d}} \prod_{a=1}^{n-1} \iint_M \mathrm{d}^d z_a\ \mathrm{d}^d z'_a \iint_{M^*} \mathrm{d}^d k_a\ \mathrm{d}^d k'_a\ \mathrm{e}^{-\,\mathrm{i}\,(k_a\cdot z'_a - k'_a\cdot z_a)} \\
&\qquad \times\ \mathrm{e}^{\,\mathrm{i}\,(k_1+\cdots+k_{n-1}-k'_{n-1})\cdot x}\ \mathrm{e}^{-\,\mathrm{i}\,k'_{n-1}\cdot(z_1+\cdots+z_{n-2})} \\
&\qquad \times \mathrm{e}^{\,\mathrm{i}\,k_a\cdot(z_1+\cdots+z_{a-1})} \prod_{b=2}^{n} f_b(z'_{n-b+1})\ f_1(z_{n-1}\,) \\
&\qquad\qquad \times\ \exp\Big(-\frac{\mathrm{i}\,\ell_s^4}{12} \sum_{c=1}^{n-1} R(k_c,k'_c,k_1+\cdots+k_{c-1})\Big) \\
&\qquad\qquad\qquad \times\ \exp\Big(-\frac{\mathrm{i}\,\ell_s^4}{6\hbar} \sum_{c=1}^{n-1} \theta_{\bar p}(k_c,k'_c\,)\Big)\ ,
\end{aligned} \tag{3.11}
$$

where we relabelled $z'_{n-b+1} \to z'_{n-b+1} - (x+z_1+\cdots+z_{n-b})$ for $2\le b\le n$, and $z_{n-1} \to z_{n-1} - (x+z_1+\cdots+z_{n-2})$. Integrating over z_a for $a=1,\dots,n-2$ gives delta-function constraints

$$
k'_a = k'_{n-1} - (k_{a+1}+\cdots+k_{n-1}) \qquad \text{for} \quad a=1,\dots,n-2\ . \tag{3.12}
$$

Integrating over z'_a for $a=1,\dots,n-1$ and over z_{n-1}, and after relabelling $k_1\to k_n$, $k'_{n-1}\to -k_1$ and $k'_{n-b+1}\to k_b$ for $b=2,\dots,n$, we then get

$$
\begin{aligned}
\mu_{\bar p}^{(n)}(f_1\otimes\cdots\otimes f_n)(x) &= \prod_{a=1}^{n} \int_{M^*} \mathrm{d}^d k_a\ \hat f_a(k_a)\ \mathrm{e}^{\,\mathrm{i}\,(k_1+\cdots+k_n)\cdot x} \\
&\qquad \times\ \exp\Big(\frac{\mathrm{i}\,\ell_s^4}{12} \sum_{c=1}^{n-1} R(k_{n-c+1},k_1+\cdots+k_{n-c},k_{n-c+2}+\cdots+k_n)\Big) \\
&\qquad\qquad \times\ \exp\Big(\frac{\mathrm{i}\,\ell_s^4}{6\hbar} \sum_{c=1}^{n-1} \theta_{\bar p}(k_{n-c+1},k_1+\cdots+k_{n-c}\,)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{a=1}^{n} \int_{M^*} \mathrm{d}^d k_a \ \hat{f}_a(k_a) \ \mathrm{e}^{\,\mathrm{i}\,(k_1+\cdots+k_n)\cdot x} \\
&\qquad \times \ \exp\Big(-\frac{\mathrm{i}\,\ell_s^4}{12} \sum_{c=1}^{n-1} \sum_{a=1}^{n-c} \ \sum_{b=n-c+2}^{n} R(k_a, k_{n-c+1}, k_b)\Big) \\
&\qquad\qquad \times \ \exp\Big(-\frac{\mathrm{i}\,\ell_s^4}{6\hbar} \sum_{c=1}^{n-1} \sum_{a=1}^{n-c} \theta_{\bar p}(k_a, k_{n-c+1}\,)\Big)
\end{aligned}
\tag{3.13}
$$

where we used multilinearity and antisymmetry of the trivector and bivector terms. Rewriting the summations we finally arrive at

$$
\begin{aligned}
\mu_{\bar p}^{(n)}(f_1\otimes\cdots\otimes f_n)(x) \ &= \ \prod_{a=1}^{n} \int_{M^*} \mathrm{d}^d k_a \ \hat{f}_a(k_a) \ \mathrm{e}^{\,\mathrm{i}\,(k_1+\cdots+k_n)\cdot x} \\
&\qquad \times \ \exp\Big(-\frac{\mathrm{i}\,\ell_s^4}{12} \sum_{1\le a<e<b\le n} R(k_a,k_e,k_b)\Big) \\
&\qquad\qquad \times \ \exp\Big(-\frac{\mathrm{i}\,\ell_s^4}{6\hbar} \sum_{1\le a<b\le n} \theta_{\bar p}(k_a,k_b)\Big) \ ,
\end{aligned}
\tag{3.14}
$$

which is the asserted result written using the momentum space representation for the $\star_{\bar p}$ product and the R-flux differential operators. $\square$

Besides the fundamental 2-products $\mu_{\bar p}^{(2)}$, which we have seen coincide with the Moyal–Weyl star products $\star_{\bar p}$, the basic triproducts are given by

$$
\mu_{\bar p}^{(3)}(f\otimes g\otimes h) = \star_{\bar p}\Big(\exp\big(\tfrac{\ell_s^4}{12}\,R^{ijk}\,\partial_i\otimes\partial_j\otimes\partial_k\,\big)(f\otimes g\otimes h)\Big) \ . \tag{3.15}
$$

The coordinate space commutator in (2.1) is then reproduced by the commutator bracket for $\mu_{\bar p}^{(2)}$, while the 3-bracket (2.2) is reproduced by $[x^i,x^j,x^k]_{\bar p}$, where

$$
[f_1,f_2,f_3]_{\bar p} := \sum_{\sigma\in S_3} (-1)^{|\sigma|}\,\mu_{\bar p}^{(3)}(f_{\sigma(1)}\otimes f_{\sigma(2)}\otimes f_{\sigma(3)}) \tag{3.16}
$$

with S_3 the group of permutations of the set $\{1,2,3\}$. The general n-triproducts satisfy the reduction properties

$$
\mu_{\bar p}^{(n)}\big(f_1\otimes\cdots\otimes(f_i\!=\!1)\otimes\cdots\otimes f_n\big) \ = \ \mu_{\bar p}^{(n-1)}\big(f_1\otimes\cdots\otimes\widehat{f_i}\otimes\cdots\otimes f_n\big) \tag{3.17}
$$

where $\widehat{f_i}$ denotes omission of f_i, $i=1,\dots,n$; these reductions consistently yield

$$
\mu_{\bar p}^{(3)}(f\otimes g\otimes 1) = \mu_{\bar p}^{(3)}(f\otimes 1\otimes g) = \mu_{\bar p}^{(3)}(1\otimes f\otimes g) = \mu_{\bar p}^{(2)}(f\otimes g) = f\star_{\bar p} g \ . \tag{3.18}
$$

However, for $n>2$ the n-triproducts $\mu_{\bar p}^{(n)}$ cannot be defined by iteration from m-triproducts with $m<n$. This is in contrast to the precursor definition (3.7) in terms of phase space star products, and also to the n-star products which are defined in (3.8) by iteration of the Moyal–Weyl star products.

One of the most distinctive features of the n-triproducts is their trivialization on-shell, i.e., after integration over M.

Proposition 3.19. $\displaystyle\int_M \mathrm{d}^d x\ \mu_{\bar p}^{(n)}(f_1\otimes\cdots\otimes f_n) = \int_M \mathrm{d}^d x\ f_1\star_{\bar p}\cdots\star_{\bar p} f_n\ .$

Proof. Using the momentum space integral formula (3.14) for the n-triproduct, we have

$$\begin{aligned}\int_M \mathrm{d}^d x\ \mu_{\bar p}^{(n)}(f_1\otimes\cdots\otimes f_n) &= (2\pi)^d\ \prod_{a=1}^n\ \int_{M^*} \mathrm{d}^d k_a\ \hat f_a(k_a)\ \delta(k_1+\cdots+k_n)\\ &\qquad \times\ \exp\Big(-\frac{\mathrm{i}\,\ell_s^4}{12}\sum_{1\le a<b<c\le n} R(k_a,k_b,k_c)\Big) \\ &\qquad \times\ \exp\Big(-\frac{\mathrm{i}\,\ell_s^4}{6\hbar}\sum_{1\le a<b\le n}\theta_{\bar p}(k_a,k_b)\Big)\ ,\end{aligned} \tag{3.20}$$

where the delta-function enforcing momentum conservation arises by translation-invariance on M of the definition (3.7). Using multilinearity of the trivector R we have

$$\sum_{1\le a<b<c\le n} R(k_a,k_b,k_c) = \sum_{b=2}^{n-1}\ \sum_{a=1}^{b-1}\ \sum_{c=b+1}^{n} R(k_a,k_b,k_c) = \sum_{b=2}^{n-1} R\Big(\sum_{a=1}^{b-1}\ k_a\ ,\ k_b\ ,\ \sum_{c=b+1}^{n}\ k_c\Big)\ , \tag{3.21}$$

and each term in the sum over b vanishes by antisymmetry of R after imposing momentum conservation $\sum_{a=1}^n\ k_a=0$. □

From Proposition 3.19 it follows that the leaf of zero momentum is singled out by the feature that on-shell there are no signs of noncommutativity or nonassociativity. On the special slice $\bar p=0$ the 2-product

$$\mu_0^{(2)}(f\otimes g) = f\,g \tag{3.22}$$

is the ordinary multiplication of functions, while for $n\ge 3$ the expression for the n-triproduct from Proposition 3.9 becomes

$$\mu_0^{(n)}(f_1\otimes\cdots\otimes f_n) = \mu\Big[\exp\Big(\frac{\ell_s^4}{12}\sum_{1\le a<b<c\le n} R^{ijk}\,\partial_i^a\,\partial_j^b\,\partial_k^c\Big)(f_1\otimes\cdots\otimes f_n)\Big]\ . \tag{3.23}$$

These products coincide with the sequence of n-triproducts for $n\ge 2$ that was proposed by [12] from an analysis of off-shell closed string tachyon amplitudes in the toroidal flux model to linear order in the flux components R^{ijk}. By Proposition 3.19 they obey the on-shell condition

$$\int_M \mathrm{d}^d x\ \mu_0^{(n)}(f_1\otimes\cdots\otimes f_n) = \int_M \mathrm{d}^d x\ f_1\cdots f_n\ , \tag{3.24}$$

and hence all traces of nonassociativity disappear in closed string scattering amplitudes. Here we have reproduced this conjectural triproduct to all orders in R^{ijk} from the all orders phase space star product derived by [26]. The correspondence between the phase space star product restricted to functions on M and the triproducts of [12] was already noted by [26] for $n=2,3$ (see also [6]); here we have extended and generalised the correspondence to all $n>3$.

The triproducts for $\bar p\neq 0$ lead to on-shell correlation functions which are cyclically invariant, by cyclicity of the Moyal–Weyl products $\star_{\bar p}$ for all $\bar p$. In this case the scattering amplitudes resemble those of *open* strings on a D-brane with two-form B-field inverse to the bivector $\theta_{\bar p}$. The appearence of the more general triproducts $\mu_{\bar p}^{(n)}$ is natural from the perspective of the

open/closed string duality described in [26, Section 2.4], which is a crucial ingredient in the derivation of the nonassociative phase space star product from closed string correlation functions; in fact, in [24, Appendix] it is argued that closed string momentum and winding modes define a certain notion of D-brane in closed string theory. Moreover, they are natural in light of the momentum space noncommutative gerbe structure of the nonassociative phase space description of the parabolic R-flux background and the closed string Seiberg–Witten maps relating associative and nonassociative theories, as described in [26, Section 3.4]. Hence in the following we work with the general leaves of constant momentum in order to retain as much of the (precursor) phase space nonassociative geometry on M as possible in our analysis, with the understanding that bonafide closed string scattering amplitudes require setting $\bar{p} = 0$ at the end of the day. We can interpret this distinction in the following way: If we regard the nonassociative field theory constructed on phase space as the fundamental field theory of closed strings in the non-geometric R-flux frame (as was done in [26] and as we do throughout in the present paper), with the leaves of constant momentum being the on-shell physical sectors, then integrating over momenta traces out extra degrees of freedom leading to violation of associativity. From this perspective localizing on the $\bar{p} = 0$ leaf corresponds to working on a closed string vacuum as a lowest order approximation to the theory, while higher (M2-brane) excitations involve fluctuations out of the $\bar{p} = 0$ leaf and probe the whole structure of the full nonassociative field theory.

3.2 Association relations

In our definition (3.7) we used a particular bracketing for the star product of n functions on phase space. As this star product is nonassociative, it is natural to ask what happens when one chooses different orderings. It was shown by [27] that the different choices of associating the functions is controlled by an associator, which can be described by a tridifferential operator

$$\Phi = \exp\left(\tfrac{\ell_s^4}{6}\, R^{ijk}\, \partial_i \otimes \partial_j \otimes \partial_k\right) . \tag{3.25}$$

The associator relates the two different products of three functions on phase space $(\,\underline{f} \star \underline{g}\,) \star \underline{h}$ and $\underline{f} \star (\,\underline{g} \star \underline{h}\,)$. If we introduce the notation

$$\Phi(\,\underline{f} \otimes \underline{g} \otimes \underline{h}\,) =: \underline{f}^{\phi} \otimes \underline{g}^{\phi} \otimes \underline{h}^{\phi} \tag{3.26}$$

where summation over the index ϕ is understood, then the relation is

$$(\,\underline{f} \star \underline{g}\,) \star \underline{h} = \underline{f}^{\phi} \star (\,\underline{g}^{\phi} \star \underline{h}^{\phi}\,) . \tag{3.27}$$

We can write this relation more algebraically as

$$\mu_\star \circ (\mu_\star \otimes \mathrm{id}) = \mu_\star \circ (\mathrm{id} \otimes \mu_\star) \circ \Phi , \tag{3.28}$$

where $\mu_\star(\,\underline{f} \otimes \underline{g}\,) := \underline{f} \star \underline{g}$ with the linear maps

$$\mu_\star \circ (\mu_\star \otimes \mathrm{id}) : (\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A} \longrightarrow \mathcal{A} \qquad \text{and} \qquad \mu_\star \circ (\mathrm{id} \otimes \mu_\star) : \mathcal{A} \otimes (\mathcal{A} \otimes \mathcal{A}) \longrightarrow \mathcal{A} . \tag{3.29}$$

Here we regard the associator as a linear map

$$\Phi : (\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes (\mathcal{A} \otimes \mathcal{A}) \qquad \text{with} \quad (\,\underline{f} \otimes \underline{g}\,) \otimes \underline{h} \xmapsto{\ \Phi\ } \underline{f}^{\phi} \otimes (\,\underline{g}^{\phi} \otimes \underline{h}^{\phi}\,) , \tag{3.30}$$

where the parentheses emphasize that on the left-hand side the product $\mu_\star \circ (\mu_\star \otimes \mathrm{id})$ naturally acts while on the right-hand side the product $\mu_\star \circ (\mathrm{id} \otimes \mu_\star)$ acts. Applying these natural products to (3.30) yields the identity (3.27).

The operator Φ is a 3-cocycle in the Hopf algebra of translations and Bopp shifts in phase space $\mathcal{M}$ that we will discuss in Section 4; this means that it obeys the pentagon relations which states that the two possible ways of reordering the brackets from left to right $\big((\,\mathcal{A} \otimes \mathcal{A}\,) \otimes \mathcal{A}\,\big) \otimes \mathcal{A} \to \mathcal{A} \otimes \big(\,\mathcal{A} \otimes (\,\mathcal{A} \otimes \mathcal{A}\,)\big)$ are equivalent, i.e., the diagram

$$\begin{tikzcd}
& (\,\mathcal{A} \otimes \mathcal{A}\,) \otimes (\,\mathcal{A} \otimes \mathcal{A}\,) \arrow[dr, "\Phi_{\mathcal{A}\,,\,\mathcal{A}\,,\,\mathcal{A} \otimes \mathcal{A}}"] & \\
\big((\,\mathcal{A} \otimes \mathcal{A}\,) \otimes \mathcal{A}\,\big) \otimes \mathcal{A} \arrow[ur, "\Phi_{\mathcal{A} \otimes \mathcal{A}\,,\,\mathcal{A}\,,\,\mathcal{A}}"] \arrow[d, "\Phi_{\mathcal{A}\,,\,\mathcal{A}\,,\,\mathcal{A}} \otimes \mathrm{id}"'] & & \mathcal{A} \otimes \big(\,\mathcal{A} \otimes (\,\mathcal{A} \otimes \mathcal{A}\,)\big) \\
\big(\,\mathcal{A} \otimes (\,\mathcal{A} \otimes \mathcal{A}\,)\big) \otimes \mathcal{A} \arrow[rr, "\Phi_{\mathcal{A}\,,\,\mathcal{A} \otimes \mathcal{A}\,,\,\mathcal{A}}"] & & \mathcal{A} \otimes \big((\,\mathcal{A} \otimes \mathcal{A}\,) \otimes \mathcal{A}\,\big) \arrow[u, "\mathrm{id} \otimes \Phi_{\mathcal{A}\,,\,\mathcal{A}\,,\,\mathcal{A}}"']
\end{tikzcd} \tag{3.31}$$

commutes. This means that we can rewrite the products $((\,\underline{f} \star \underline{g}\,) \star \underline{h}\,) \star \underline{k}$ as linear combinations of products of the kind $\underline{f}' \star (\,\underline{g}' \star (\,\underline{h}' \star \underline{k}'\,))$. There are *a priori* two different linear combinations, respectively obtained by following the upper and lower paths in the diagram (3.31). Commutativity of the diagram asserts that these two paths are equivalent. The action of the differential operator $\Phi_{\mathcal{A} \otimes \mathcal{A}\,,\,\mathcal{A}\,,\,\mathcal{A}}$ on $(\,\underline{f} \otimes \underline{g}\,) \otimes \underline{h} \otimes \underline{k}$ is inherited from the Leibniz rule $\partial_i(\,\underline{f} \otimes \underline{g}\,) = \partial_i\,\underline{f} \otimes \underline{g} + \underline{f} \otimes \partial_i\,\underline{g}$ and it reads as

$$\Phi_{\mathcal{A} \otimes \mathcal{A}\,,\,\mathcal{A}\,,\,\mathcal{A}}\big((\,\underline{f} \otimes \underline{g}\,) \otimes \underline{h} \otimes \underline{k}\,\big) = \exp\big(\tfrac{\ell_s^4}{6}\,R^{ijk}\,(\partial_i^1\,\partial_j^3\,\partial_k^4 + \partial_i^2\,\partial_j^3\,\partial_k^4)\big)(\,\underline{f} \otimes \underline{g} \otimes \underline{h} \otimes \underline{k}\,)\ . \tag{3.32}$$

If we apply the product $\mu_\star \circ (\mu_\star \otimes \mu_\star)$ to this expression we obtain

$$\mu_\star \circ (\mu_\star \otimes \mu_\star) \circ \Phi_{\mathcal{A} \otimes \mathcal{A}\,,\,\mathcal{A}\,,\,\mathcal{A}}\big((\,\underline{f} \otimes \underline{g}\,) \otimes \underline{h} \otimes \underline{k}\,\big) = \big((\,\underline{f} \star \underline{g}\,) \star \underline{h}\,\big) \star \underline{k}\ , \tag{3.33}$$

as is read off from the upper arrow in (3.31). The differential operators $\Phi_{\mathcal{A}\,,\,\mathcal{A}\,,\,\mathcal{A} \otimes \mathcal{A}}$ and $\Phi_{\mathcal{A}\,,\,\mathcal{A} \otimes \mathcal{A}\,,\,\mathcal{A}}$ are similarly defined.

The associator Φ also relates the two inequivalent triproducts of functions on M given by $s_{\bar p}^*\big[(\pi^*f \star \pi^*g) \star \pi^*h\big]$ and $s_{\bar p}\big[\pi^*f \star (\pi^*g \star \pi^*h)\big]$. In order to present an explicit expression for the product $s_{\bar p}\big[\pi^*f \star (\pi^*g \star \pi^*h)\big]$ we observe that there is an obvious one-to-one correspondence between differential operators ∂_i on $\mathcal{A} = C^\infty(\mathcal{M})$ and on $A = C^\infty(M)$, and that for any function f on M one has $s_{\bar p}(\partial_i(\pi^*f)) = \partial_i f$. This implies that the associator Φ naturally acts as a differential operator on $A \otimes A \otimes A$. We define

$$\Phi^{-1}(\,\underline{f} \otimes \underline{g} \otimes \underline{h}\,) = \underline{f}^{\bar\phi} \otimes \underline{g}^{\bar\phi} \otimes \underline{h}^{\bar\phi} \qquad \text{and} \qquad \Phi^{-1}(f \otimes g \otimes h) = f^{\bar\phi} \otimes g^{\bar\phi} \otimes h^{\bar\phi} \tag{3.34}$$

with implicit summation as before, and then we find

$$\begin{aligned}
s_{\bar p}^*\big[\big(\pi^*f \star (\pi^*g \star \pi^*h)\big)\big] &= s_{\bar p}^*\big[\big((\pi^*f)^{\bar\phi} \star (\pi^*g)^{\bar\phi}\big) \star (\pi^*h)^{\bar\phi}\,\big] \\
&= s_{\bar p}^*\big[\big((\pi^*f^{\bar\phi}) \star (\pi^*g^{\bar\phi})\big) \star (\pi^*h^{\bar\phi}\,)\big]
\end{aligned}$$

$$
\begin{aligned}
&= \mu^{(3)}_{\bar p}\big(f^{\bar\phi}\otimes g^{\bar\phi}\otimes h^{\bar\phi}\big) \\
&= \star_{\bar p}\Big(\exp\big(\tfrac{\ell_s^4}{12}\,R^{ijk}\,\partial_i\otimes\partial_j\otimes\partial_k\big)\,\Phi^{-1}(f\otimes g\otimes h)\Big) \\
&= \star_{\bar p}\Big(\exp\big(-\tfrac{\ell_s^4}{12}\,R^{ijk}\,\partial_i\otimes\partial_j\otimes\partial_k\big)(f\otimes g\otimes h)\Big)\ .
\end{aligned}
\tag{3.35}
$$

Thus we can generate the new triproduct $s^*_{\bar p}\big[\big(\pi^*f\star(\pi^*g\star\pi^*h)\big)\big]$ of fields by applying the inverse of the associator Φ directly to functions on M as in (3.35). This new triproduct also obeys Proposition 3.19, since from (3.35) we have

$$
\int_M \mathrm{d}^dx\ s^*_{\bar p}\big[\big(\pi^*f\star(\pi^*g\star\pi^*h)\big)\big] = \int_M \mathrm{d}^dx\ \mu^{(3)}_{\bar p}\big[\Phi^{-1}(f\otimes g\otimes h)\big] = \int_M \mathrm{d}^dx\ f\star_{\bar p}g\star_{\bar p}h\ . \tag{3.36}
$$

A quick way to prove this is to note that

$$
\Phi^{-1}\big(\mathrm{e}^{\,\mathrm{i}\,k_1\cdot x}\otimes\mathrm{e}^{\,\mathrm{i}\,k_2\cdot x}\otimes\mathrm{e}^{\,\mathrm{i}\,k_3\cdot x}\big) = \mathrm{e}^{\frac{\mathrm{i}\,\ell_s^4}{6}\,R(k_1,k_2,k_3)}\ \big(\mathrm{e}^{\,\mathrm{i}\,k_1\cdot x}\otimes\mathrm{e}^{\,\mathrm{i}\,k_2\cdot x}\otimes\mathrm{e}^{\,\mathrm{i}\,k_3\cdot x}\big)\ , \tag{3.37}
$$

and the extra phase factor is unity after imposing momentum conservation $k_1+k_2+k_3=0$.

We can now generate all induced 4-triproducts from the diagram (3.31). For example one has

$$
\begin{aligned}
s^*_{\bar p}\big[(\pi^*f\star\pi^*g)\star(\pi^*h\star\pi^*k)\big] &= \mu^{(4)}_{\bar p}\big[\Phi^{-1}_{A\otimes A\,,\,A\,,\,A}\big((f\otimes g)\otimes h\otimes k\big)\big] \\
&= \mu^{(4)}_{\bar p}\big[\exp\big(-\tfrac{\ell_s^4}{6}\,R^{ijk}\,(\partial_i^1\,\partial_j^3\,\partial_k^4+\partial_i^2\,\partial_j^3\,\partial_k^4)\big)(f\otimes g\otimes h\otimes k)\big)\big] \\
&= \star_{\bar p}\big[\exp\big(\tfrac{\ell_s^4}{12}\,R^{ijk}\,(\partial_i^1\,\partial_j^2\,\partial_k^3+\partial_i^1\,\partial_j^2\,\partial_k^4 \\
&\qquad -\partial_i^1\,\partial_j^3\,\partial_k^4-\partial_i^2\,\partial_j^3\,\partial_k^4)\big)(f\otimes g\otimes h\otimes k)\big]\ .
\end{aligned}
\tag{3.38}
$$

A completely analogous calculation following both upper arrows in the diagram (3.31) gives

$$
\begin{aligned}
s^*_{\bar p}\big[\pi^*f\star\big(\pi^*g\star(\pi^*h\star\pi^*k)\big)\big] &= \mu^{(4)}_{\bar p}\big[\Phi^{-1}_{A\otimes A\,,\,A\,,\,A}\,\Phi^{-1}_{A\,,\,A\,,\,A\otimes A}\big(f\otimes g\otimes(h\otimes k)\big)\big] \\
&= \star_{\bar p}\Big[\exp\Big(-\frac{\ell_s^4}{12}\,R^{ijk}\sum_{1\le a<b<c\le 4}\partial_i^a\,\partial_j^b\,\partial_k^c\Big)(f\otimes g\otimes h\otimes k)\Big]\ .
\end{aligned}
\tag{3.39}
$$

These new 4-triproducts serve just as well for describing the products among off-shell closed string tachyon vertex operators, and indeed we find as in Proposition 3.19 the on-shell result

$$
\begin{aligned}
\int_M \mathrm{d}^dx\ s^*_{\bar p}\big[(\pi^*f\star\pi^*g)\star(\pi^*h\star\pi^*k)\big] &= \int_M \mathrm{d}^dx\ \mu^{(4)}_{\bar p}\big[\Phi^{-1}_{A\otimes A\,,\,A\,,\,A}\big((f\otimes g)\otimes h\otimes k\big)\big] \\
&= \int_M \mathrm{d}^dx\ \star_{\bar p}\big[\Phi^{-1}_{A\otimes A\,,\,A\,,\,A}\big((f\otimes g)\otimes h\otimes k\big)\big] \\
&= \int_M \mathrm{d}^dx\ \star_{\bar p}\big[\Phi^{-1}\big((f\star_{\bar p}g)\otimes h\otimes k\big)\big] \\
&= \int_M \mathrm{d}^dx\ f\star_{\bar p}g\star_{\bar p}h\star_{\bar p}k
\end{aligned}
\tag{3.40}
$$

where in the last equality we used (3.36). Letting $R^{ijk}\to -R^{ijk}$ in Proposition 3.19 we also immediately see that the integral of (3.39) coincides with (3.40).

The situation is notably different if one follows the left vertical arrow in the diagram (3.31). We first compute the product

$$\begin{aligned} s_{\bar p}^*\big[\big(\pi^*f\star(\pi^*g\star\pi^*h)\big)\star\pi^*k\big] &= \mu_{\bar p}^{(4)}\big[\Phi^{-1}_{A\,,\,A\otimes A\,,\,A}\big(f\otimes(g\otimes h)\otimes k\big)\big] \\ &= \star_{\bar p}\Big[\exp\big(\tfrac{\ell_s^4}{12}\,R^{ijk}\,(-\partial_i^1\,\partial_j^2\,\partial_k^3+\partial_i^1\,\partial_j^2\,\partial_k^4 \\ &\qquad +\partial_i^1\,\partial_j^3\,\partial_k^4+\partial_i^2\,\partial_j^3\,\partial_k^4)\big)(f\otimes g\otimes h\otimes k)\Big]\ . \end{aligned} \tag{3.41}$$

Analogously we find for the product in the bottom right corner of (3.31) as

$$\begin{aligned} s_{\bar p}^*\big[\pi^*f\star\big((\pi^*g\star\pi^*h)\star\pi^*k\big)\big] &= \mu_{\bar p}^{(4)}\big[(\Phi^{-1}\otimes\mathrm{id})\,\Phi^{-1}_{A\,,\,A\otimes A\,,\,A}\big(f\otimes(g\otimes h)\otimes k\big)\big] \\ &= \star_{\bar p}\Big[\exp\big(\tfrac{\ell_s^4}{12}\,R^{ijk}\,(-\partial_i^1\,\partial_j^2\,\partial_k^3-\partial_i^1\,\partial_j^2\,\partial_k^4 \\ &\qquad -\,\partial_i^1\,\partial_j^3\,\partial_k^4+\partial_i^2\,\partial_j^3\,\partial_k^4)\big)(f\otimes g\otimes h\otimes k)\Big]\ , \end{aligned} \tag{3.42}$$

and it is easy to see that a further application of $\mathrm{id}\otimes\Phi^{-1}$ leads exactly to the 4-triproduct (3.39). As previously the application of $\Phi^{-1}_{A\,,\,A\otimes A\,,\,A}$ does not alter closed string amplitudes, but the application of $\Phi^{-1}\otimes\mathrm{id}$ does, and we find

$$\begin{aligned} \int_M \mathrm{d}^dx\ s_{\bar p}^*\big[\big(\pi^*f\star(\pi^*g\star\pi^*h)\big)\star\pi^*k\big] &= \int_M \mathrm{d}^dx\ \mu_{\bar p}^{(4)}\big[\Phi^{-1}(f\otimes g\otimes h)\otimes k\big] \\ &= \int_M \mathrm{d}^dx\ \mu_{\bar p}^{(4)}\big(f^{\bar\phi}\otimes g^{\bar\phi}\otimes h^{\bar\phi}\otimes k\big) \\ &= \int_M \mathrm{d}^dx\ s_{\bar p}^*\big[\pi^*f\star\big((\pi^*g\star\pi^*h)\star\pi^*k\big)\big]\ . \end{aligned} \tag{3.43}$$

In general one has

$$\begin{aligned} \int_M \mathrm{d}^dx\ \mu_{\bar p}^{(4)}\big(f^{\bar\phi}\otimes g^{\bar\phi}\otimes h^{\bar\phi}\otimes k\big) &= \int_M \mathrm{d}^dx\ f^{\bar\phi}\star_{\bar p}g^{\bar\phi}\star_{\bar p}h^{\bar\phi}\star_{\bar p}k \\ &\neq \int_M \mathrm{d}^dx\ f\star_{\bar p}g\star_{\bar p}h\star_{\bar p}k\ . \end{aligned} \tag{3.44}$$

As a consequence these two 4-triproducts leave traces of nonassociativity through additional interaction terms, and should hence not be considered as physically viable products among off-shell closed string tachyon vertex operators.

This line of reasoning can be extended to all n-triproducts with $n>4$. By MacLane's coherence theorem, all possible bracketings of star products of n functions on $\mathcal{M}$ are related by successive applications of the inverse associator Φ^{-1} to tensor products of these n functions. In all there are C_{n-1} star products, where C_n is the Catalan number of degree n. Of these there are $n-1$ triproducts which can serve as physical products among closed string vertex operators. They are obtained from the bracketings

$$\big((\cdots((\,\underline{f}_{\,1}\star\underline{f}_{\,2})\star\underline{f}_{\,3})\star\cdots)\star\underline{f}_{\,r}\big)\star\big(\underline{f}_{\,r+1}\star(\underline{f}_{\,r+2}\star(\underline{f}_{\,r+3}\star(\cdots\star\underline{f}_{\,n})\cdots))\big) \tag{3.45}$$

for $1\leq r\leq n-1$, which for $r\leq n-2$ are obtained by successive applications for $s=r,r+1,\dots,n-3$ of the inverse associators

$$\Phi^{-1}_{(\cdots((\mathcal{A}\otimes\mathcal{A})\otimes\mathcal{A})\otimes\cdots)\otimes\mathcal{A}\,,\,\mathcal{A}\,,\,\mathcal{A}\otimes(\mathcal{A}\otimes(\mathcal{A}\otimes(\cdots\otimes\mathcal{A})\cdots))} \tag{3.46}$$

to $((\cdots(\,\underline{f}_1\otimes\,\underline{f}_2)\otimes\,\underline{f}_3)\otimes\cdots)\otimes\,\underline{f}_r)\otimes\underline{f}_{r+1}\otimes(\underline{f}_{r+2}\otimes(\underline{f}_{r+3}\otimes(\cdots\otimes\underline{f}_n)\cdots)))$, where the first slot of Φ^{-1} in (3.46) contains s tensor products of the algebra $\mathcal{A}=C^\infty(\mathcal{M})$. The action of the inverse associator (3.46) is again obtained using the Leibniz rule for the partial derivatives ∂_i on $\mathcal{A}^{\otimes s}$ and $\mathcal{A}^{\otimes n-s-1}$. It follows that for the specific associator we are considering there is no ambiguity in rewriting the expression (3.46) simply as $\Phi^{-1}_{\mathcal{A}^{\otimes s},\,\mathcal{A}\,,\,\mathcal{A}^{\otimes n-s-1}}$. Its momentum space representation is obtained by applying it to the tensor products of plane waves

$$\begin{aligned}\big[(\cdots(\,\mathrm{e}^{\,\mathrm{i}\,k_1\cdot x}\otimes\mathrm{e}^{\,\mathrm{i}\,k_2\cdot x})\otimes\cdots)\otimes\mathrm{e}^{\,\mathrm{i}\,k_s\cdot x}\big]\ &\otimes\ \mathrm{e}^{\,\mathrm{i}\,k_{s+1}\cdot x}\\ &\otimes\ \big[\mathrm{e}^{\,\mathrm{i}\,k_{s+2}\cdot x}\otimes\big(\mathrm{e}^{\,\mathrm{i}\,k_{b+3}\cdot x}\otimes(\cdots\otimes\mathrm{e}^{\,\mathrm{i}\,k_n\cdot x})\cdots\big)\big]\end{aligned}\tag{3.47}$$

which yields the phase factor

$$\exp\big(\tfrac{\mathrm{i}\,\ell_s^4}{6}\,R(k_1+\cdots+k_s\,,\,k_{s+1}\,,\,k_{s+2}+\cdots+k_n)\big)\ .\tag{3.48}$$

The n-triproducts (3.45) on phase space $\mathcal{M}$ induce triproducts on configuration space M by setting $\underline{f}_i=\pi^*f_i$ for $i=1,\dots,n$ and then pulling these products back along the local sections $s_{\bar p}:M\to\mathcal{M}$. Their explicit expressions in terms of the Moyal-Weyl product and of the R-flux tridifferential operators is obtained by replacing the tridifferential operators of Proposition 3.9 with

$$\mathrm{e}^{\,R_{(r,n)}}:=\exp\Bigg(\frac{\ell_s^4}{12}\sum_{1\le a<b<c\le n}R^{ijk}\,\partial_i^a\,\partial_j^b\,\partial_k^c-\frac{\ell_s^4}{6}\sum_{b=r+1}^{n-1}\ \sum_{1\le a<b<c\le n}R^{ijk}\,\partial_i^a\,\partial_j^b\,\partial_k^c\Bigg)\ .\tag{3.49}$$

These n-triproducts also obey the on-shell condition of Proposition 3.19 (as can be seen from the vanishing of the phase in (3.48) after integrating over M which yields momentum conservation $k_1+\cdots+k_n=0$).

The remaining triproducts violate the on-shell condition because they are obtained from the previous ones by applying those inverse associators that like the vertical arrows in (3.31) do not act on all the tensor products entries; in this case total momentum conservation does not imply the trivialization of their action. For example one has

$$\begin{aligned}\int_M\mathrm{d}^dx\ s_{\bar p}^*\Big[&\big(((\cdots((\,\pi^*f_1\star\pi^*f_2)\star\pi^*f_3)\star\cdots)\star\pi^*f_{r-2})\star(\pi^*f_{r-1}\star\pi^*f_r)\big)\\ &\qquad\star\big(\,\pi^*f_{r+1}\star(\,\pi^*f_{r+2}\star(\,\pi^*f_{r+3}\star(\cdots\star\pi^*f_n)\cdots))\big)\Big]\\ &=\int_M\mathrm{d}^dx\ \star_{\bar p}\Big[\,\mathrm{e}^{\,R_{(r,n)}}\,\big(\Phi^{-1}_{\mathcal{A}^{\otimes r-2},\,\mathcal{A}\,,\,\mathcal{A}}\otimes\mathrm{id}^{\otimes n-r}\big)(f_1\otimes\cdots\otimes f_n)\Big]\\ &=\int_M\mathrm{d}^dx\ \star_{\bar p}\Big[\Phi^{-1}_{\mathcal{A}^{\otimes r-2},\,\mathcal{A}\,,\,\mathcal{A}}(f_1\otimes\cdots\otimes f_r)\Big]\star_{\bar p}f_{r+1}\star_{\bar p}f_{r+2}\cdots\star_{\bar p}f_n\ .\end{aligned}\tag{3.50}$$

Because of momentum conservation, all n-triproducts which differ from that above by a sequence of inverse associators acting non-trivially on each tensor product entry (like the horizontal or diagonal arrows in (3.31)) yield the same integral (3.50). This procedure can be iterated to obtain all remaining triproducts and their on-shell associativity violating amplitudes, but we refrain from detailing further the combinatorics.

4 Triproducts from phase space cochains

4.1 Motivation: Differential geometry of nonassociative R-space

In order to extend the considerations of Section 3 to more general geometric entities, such as differential forms and vector fields, we need to uncover the more algebraic construction that underlies the explicit representation via Fourier transformation that we used so far. Such a description of the phase space nonassociative geometry was introduced by [27], while in [7] a general categorical perspective on cochain twist deformation is presented. Here we shall further develop the phase space nonassociative geometry, recover the main results of Section 3 in this algebraic context, and then study the exterior product of forms in nonassociative phase space; we further derive the induced products on the leaves of constant momentum and their property under integration. We also study the deformed Lie algebra of infinitesimal diffeomorphisms on phase space, as well as its action on exterior forms and tensor fields. These geometric structures are again induced on each constant momentum leaf and in particular in configuration space (the leaf of vanishing momentum). We thus derive the configuration space geometry of exterior forms and vector fields, together with their deformed Lie algebra and Jacobiator, at all orders in the R-flux components R^{ijk} from the geometry of phase space that we canonically construct via 2-cochain twist deformation.

4.2 Cochain twist deformations and nonassociative star products

Consider the Lie algebra $\mathfrak{h}$ with generators $P_i, \tilde{P}^i, M^i$ for $i = 1, \dots, d$ and relations

$$[\tilde{P}^i, M^j] = \tfrac{1}{6}\, \ell_s^4\, R^{ijk}\, P_k \ , \tag{4.1}$$

with all other Lie brackets vanishing. This Lie algebra naturally acts on functions on phase space via the representation

$$P_i(\,\underline{f}\,) := \partial_i\, \underline{f} \ , \qquad \tilde{P}^i(\,\underline{f}\,) := \mathrm{i}\, \hbar\, \tilde{\partial}^i\, \underline{f} \qquad \text{and} \qquad M^i(\,\underline{f}\,) = \frac{\mathrm{i}\, \ell_s^4}{6\hbar}\, R^{ijk}\, p_j\, \partial_k\, \underline{f} \tag{4.2}$$

for all $\underline{f} \in C^\infty(\mathcal{M})$. The operators P_i and $\tilde{P}^i$ respectively generate position and momentum translations in phase space, while M^i generate Bopp shifts. Alternatively, we can represent $\mathfrak{h}$ on the algebra $\mathsf{Diff}(M)$ of differential operators on M by

$$P_i = \mathrm{ad}_{\partial_i} \ , \qquad \tilde{P}^i = \mathrm{ad}_{x^i} \qquad \text{and} \qquad M^i = \tfrac{1}{6}\, \ell_s^4\, R^{ijk}\, \partial_j\ \mathrm{ad}_{\partial_k} \ , \tag{4.3}$$

with $\mathrm{ad}_{\partial_i}(x^j) = \delta_i^j = -\,\mathrm{ad}_{x^j}(\partial_i)$.

Let $U(\mathfrak{h})$ be the universal enveloping algebra of the Lie algebra $\mathfrak{h}$, i.e., the associative algebra of $\mathbb{C}$-linear combinations of products of the generators $P_i, \tilde{P}^i, M^i$ modulo the Lie algebra relations (4.1) as well as all other vanishing relations $[P_i, P_j] = 0, [P_i, M^j] = 0, [M^i, M^j] = 0$, etc. This associative unital algebra is a Hopf algebra with coproduct $\Delta : U(\mathfrak{h}) \to U(\mathfrak{h}) \otimes U(\mathfrak{h})$, counit $\varepsilon : U(\mathfrak{h}) \to \mathbb{C}$ and antipode $S : U(\mathfrak{h}) \to U(\mathfrak{h})$ defined on generators $X \in \{P_i, \tilde{P}^i, M^i\}$ as

$$\Delta(X) = X \otimes 1 + 1 \otimes X \ , \qquad \varepsilon(X) = 0 \qquad \text{and} \qquad S(X) = -X \ . \tag{4.4}$$

The coproduct Δ and counit ε are linear maps extended multiplicatively to all of $U(\mathfrak{h})$, while the antipode S is a linear map extended anti-multiplicatively to all of $U(\mathfrak{h})$, i.e., $S(X\,Y) =$

$S(Y)\,S(X)$. The action of $\mathfrak{h}$ on $C^\infty(\mathcal{M})$ naturally extends to an action of $U(\mathfrak{h})$ on $C^\infty(\mathcal{M})$ via

$$(X_1\,X_2\cdots X_n)(\,\underline{f}\,) := X_1\big(X_2(\cdots X_n(\,\underline{f}\,)\cdots)\big) \tag{4.5}$$

for all $X_1, X_2, \ldots, X_n \in \mathfrak{h}$ and $\underline{f} \in C^\infty(\mathcal{M})$.

The coproduct on generators X encodes the Leibniz rule $X(\,\underline{f}\ \underline{g}\,) = X(\,\underline{f}\,)\,\underline{g} + \underline{f}\,X(\,\underline{g}\,)$ for all functions $\underline{f}$, $\underline{g}$ on phase space. An equivalent expression for the Leibniz rule is $X(\,\underline{f}\ \underline{g}\,) = \mu\circ\Delta(X)(\,\underline{f}\otimes\underline{g}\,)$. Multiplicativity of the coproduct implies more generally that

$$\zeta(\,\underline{f}\ \underline{g}\,) = \mu\big(\Delta(\zeta)(\,\underline{f}\otimes\underline{g}\,)\big) \tag{4.6}$$

for all $\zeta\in U(\mathfrak{h})$, which equivalently reads $\zeta\circ\mu\,(\,\underline{f}\otimes\underline{g}\,) = \mu\circ\Delta(\zeta)(\,\underline{f}\otimes\underline{g}\,)$.

We can consider two twists in $U(\mathfrak{h})\otimes U(\mathfrak{h})$ given by[2]

$$F = \exp\big(-\tfrac{1}{2}\,(P_i\otimes\tilde P^i - \tilde P^i\otimes P_i)\big) \qquad \text{and} \qquad F' = \exp\big(-\tfrac{1}{2}\,(M^i\otimes P_i - P_i\otimes M^i)\big)\ . \tag{4.7}$$

They are both abelian cocycle twists of the Hopf algebra $U(\mathfrak{h})$ with the coalgebra structures Δ, ε, S, i.e., they are invertible and satisfy the relations

$$(F\otimes 1)\,(\Delta\otimes\mathrm{id})F \ = \ (1\otimes F)\,(\mathrm{id}\otimes\Delta)F\ , \tag{4.8}$$

$$(\varepsilon\otimes\mathrm{id})F \ = \ 1 \ = \ (\mathrm{id}\otimes\varepsilon)F\ , \tag{4.9}$$

plus the analogous relations for $F\to F'$. The first relation is a 2-cocycle condition which assures that the star products obtained from the twists are associative. The second relation is just a normalization condition. Since the bracket (4.1) is central in the Lie algebra $\mathfrak{h}$, we can use the Baker–Campbell–Hausdorff formula

$$\exp(A)\ \exp(B) = \exp\big([A,B]\big)\ \exp(B)\ \exp(A) \tag{4.10}$$

for $[A,B]$ central (and A,B in $U(\mathfrak{h})\otimes U(\mathfrak{h})$ and then in $U(\mathfrak{h})\otimes U(\mathfrak{h})\otimes U(\mathfrak{h})$) to compute

$$F\,F' = F'\,F \qquad \text{and} \qquad (1\otimes F'\,)\,(\mathrm{id}\otimes\Delta)F = \mathrm{e}^{\,R}\,(\mathrm{id}\otimes\Delta)F\,(1\otimes F'\,) \tag{4.11}$$

where we have introduced the central element in $U(\mathfrak{h})\otimes U(\mathfrak{h})\otimes U(\mathfrak{h})$ given by

$$\mathrm{e}^{\,R} := \exp\big(\tfrac{\ell_s^4}{12}\,R^{ijk}\,P_i\otimes P_j\otimes P_k\big)\ . \tag{4.12}$$

Note that, in the representation (4.2) on $C^\infty(\mathcal{M})$, the square of this operator is the associator (3.25), i.e.,

$$\Phi = \mathrm{e}^{\,2R}\ . \tag{4.13}$$

The n-triproducts of functions from Section 3 can also be obtained from more algebraic considerations using the twists F and F' of the Hopf algebra $U(\mathfrak{h})$. On one hand this requires a minimal amount of Hopf algebra technology, while on the other hand this approach can be applied to any algebra that carries a representation of $U(\mathfrak{h})$. In particular, we shall apply it to the algebras of exterior differential forms and of Lie derivatives (infinitesimal diffeomorphisms).

[2]The twists F and F' should be more precisely regarded respectively as power series expansions in $\hbar$ and $\ell_s^4/\hbar$, the deformation parameters that for ease of notation have been absorbed into the definition of the generators $\tilde P^i$ and M^i respectively. Then $U(\mathfrak{h})$ is an algebra over $\mathbb{C}[[\hbar,\ell_s^4/\hbar]]$, the formal power series in $\hbar$ and $\ell_s^4/\hbar$ with coefficients in $\mathbb{C}$. In this setting the twists F and F' live in a completion of the tensor product $U(\mathfrak{h})\otimes U(\mathfrak{h})$.

Let us first consider the twist F'. Associated with F' there is a new Hopf Algebra $U(\mathfrak{h})^{F'}$. The algebra structure is the same as that of $U(\mathfrak{h})$, the new coproduct is given by

$$\Delta^{F'}(\xi) = F'\,\Delta(\xi)\,F'^{-1} \tag{4.14}$$

for all $\xi \in U(\mathfrak{h})$, while $U(\mathfrak{h})^F$ has the same counit ε and antipode S as $U(\mathfrak{h})$ due to the abelian structure of the twist, i.e., P_i and M^j commute. We can now further deform the Hopf algebra $U(\mathfrak{h})^{F'}$ with the twist F. Notice that while F is a cocycle twist for the Hopf algebra $U(\mathfrak{h})$, because it satisfies (4.8) and (4.9), it is *not* a cocycle twist of the new Hopf algebra $U(\mathfrak{h})^{F'}$. To compute its failure it is convenient to compare the actions of the coproducts $\Delta^{F'}$ and Δ on the twist element F.

Proposition 4.15. $\big(\mathrm{id}\otimes\Delta^{F'}\big)F = \mathrm{e}^{\,R}\,(\mathrm{id}\otimes\Delta)F$ *and* $\big(\Delta^{F'}\otimes\mathrm{id}\big)F = \mathrm{e}^{\,-R}\,(\Delta\otimes\mathrm{id})F$.

Proof. The coproduct $\Delta^{F'}$ differs from Δ when applied to the generators $\tilde{P}^i$. We consider the term in (4.10) which is linear in B, i.e., $\exp(A)\,B\,\exp(-A) = B+[A,B]$, and we identify B with $\Delta(\tilde{P}^i)$ in order to easily compute

$$\begin{aligned}
\Delta^{F'}(\tilde{P}^i) &= F'\,\Delta(\tilde{P}^i)\,F'^{-1} \\
&= F'\,\big(\tilde{P}^i\otimes 1+1\otimes\tilde{P}^i\big)\,F'^{-1} \\
&= \tilde{P}^i\otimes 1+1\otimes\tilde{P}^i - \tfrac12\,\big[M^k\otimes P_k - P_k\otimes M^k\,,\,\tilde{P}^i\otimes 1+1\otimes\tilde{P}^i\big] \\
&= \Delta(\tilde{P}^i) - \tfrac16\,\ell_s^4\,R^{ijk}\,P_j\otimes P_k\ .
\end{aligned} \tag{4.16}$$

Next we compute

$$\begin{aligned}
\big(\mathrm{id}\otimes\Delta^{F'}\big)F &= \big(\mathrm{id}\otimes\Delta^{F'}\big)\exp\big(-\tfrac12\,(P_i\otimes\tilde{P}^i - \tilde{P}^i\otimes P_i)\big) \\
&= \exp\Big(-\tfrac12\,\big(P_i\otimes\Delta^{F'}(\tilde{P}^i) - \tilde{P}^i\otimes\Delta^{F'}(P_i)\big)\Big) \\
&= \exp\Big(-\tfrac12\,\big(P_i\otimes\Delta(\tilde{P}^i) - \tilde{P}^i\otimes\Delta(P_i)\big)\Big)\,\exp\big(\tfrac{\ell_s^4}{12}\,R^{ijk}\,P_i\otimes P_j\otimes P_k\big) \\
&= (\mathrm{id}\otimes\Delta)F\ \mathrm{e}^{\,R} \ = \ \mathrm{e}^{\,R}\,(\mathrm{id}\otimes\Delta)F\ ,
\end{aligned} \tag{4.17}$$

where in the second equality we used multiplicativity of the coproduct Δ. The proof of the second identity is very similar, or alternatively observe that $F\to F^{-1}$ under flipping of the order of its legs in the tensor product $U(\mathfrak{h})\otimes U(\mathfrak{h})$. □

As a corollary we find that F is not a cocycle twist of $U(\mathfrak{h})^{F'}$, i.e., it fails the cocycle condition for the Hopf algebra $U(\mathfrak{h})^{F'}$.

Corollary 4.18. $(1\otimes F)\,\big(\mathrm{id}\otimes\Delta^{F'}\big)F = \mathrm{e}^{\,2R}\,(F\otimes 1)\,\big(\Delta^{F'}\otimes\mathrm{id}\big)F$.

Proof. Use Proposition 4.15 to rewrite the left-hand and right-hand sides in terms of the undeformed coproduct Δ. Then recall that $\mathrm{e}^{\,R}$ is central in the algebra $U(\mathfrak{h})\otimes U(\mathfrak{h})\otimes U(\mathfrak{h})$, and finally use the cocycle property (4.8). □

The Hopf algebra $U(\mathfrak{h})$ acts on the algebra $C^\infty(\mathcal{M})$ of functions on phase space via the representation (4.2), and this action is compatible with the product in $C^\infty(\mathcal{M})$ in the sense of

(4.6). Because of this compatibility we can then deform $C^\infty(\mathcal{M})$ into a new algebra $C^\infty(\mathcal{M})_{\star_{F'}}$ defined by the new product

$$\underline{f} \star_{F'} \underline{g} = \mu_{F'}(\underline{f} \otimes \underline{g}) := \mu\big(F'^{-1}(\underline{f} \otimes \underline{g})\big) \tag{4.19}$$

for any phase space functions $\underline{f}$ and $\underline{g}$. The algebra $C^\infty(\mathcal{M})_{\star_{F'}}$ is noncommutative but associative because of the cocycle condition satisfied by F'. On $C^\infty(\mathcal{M})_{\star_{F'}}$ there is a natural action of the Hopf algebra $U(\mathfrak{h})^{F'}$; it is again defined by (4.2). The deformed coproduct $\Delta^{F'}$, which characterizes $U(\mathfrak{h})^{F'}$, describes the action of any element $\zeta \in U(\mathfrak{h})^{F'}$ on products of functions, i.e., for all $\underline{f}, \underline{g} \in C^\infty(\mathcal{M})_{\star_{F'}}$, by (4.6) one has

$$\begin{aligned}
\zeta(\underline{f} \star_{F'} \underline{g}) &= \zeta \circ \mu_{F'}(\underline{f} \otimes \underline{g}) \\
&= \zeta \circ \mu \circ F'^{-1}(\underline{f} \otimes \underline{g}) \\
&= \mu \circ \Delta(\zeta) \circ F'^{-1}(\underline{f} \otimes \underline{g}) \ = \ \mu_{F'} \circ \Delta^{F'}(\zeta)(\underline{f} \otimes \underline{g}) \ .
\end{aligned} \tag{4.20}$$

In particular it leads to a deformed Leibniz rule for the generators $\tilde{P}^i$ as can be read off from (4.16).

Now we iterate this procedure and deform the Hopf algebra $U(\mathfrak{h})^{F'}$ and the noncommutative algebra $C^\infty(\mathcal{M})_{\star_{F'}}$ with the 2-cochain $F \in U(\mathfrak{h})^{F'} \otimes U(\mathfrak{h})^{F'}$. This is equivalent to the deformation of the original Hopf algebra $U(\mathfrak{h})$ and the algebra of smooth functions $C^\infty(\mathcal{M})$ with the 2-cochain

$$\mathcal{F} := F\,F' = F'\,F \ . \tag{4.21}$$

Since F fails the $\Delta^{F'}$ 2-cocycle property, the coproduct $(\Delta^{F'})^F$, defined as

$$(\Delta^{F'})^F(\zeta) = F\,\Delta^{F'}(\zeta)\,F^{-1} = \Delta^{\mathcal{F}}(\zeta) \ , \tag{4.22}$$

will equip $(U(\mathfrak{h})^{F'})^F = U(\mathfrak{h})^{\mathcal{F}}$ with the structure of a quasi-Hopf algebra [16] (see also [27, 7]). Correspondingly, $C^\infty(\mathcal{M})_{\star_{F'}}$ deforms to a quasi-associative noncommutative algebra with product

$$\underline{f} \star_{\mathcal{F}} \underline{g} = \underline{f} \star_{F\,F'} \underline{g} = \mu_{F'}\big(F^{-1}(\underline{f} \otimes \underline{g})\big) = \mu\big((F\,F')^{-1}(\underline{f} \otimes \underline{g})\big) \tag{4.23}$$

for all phase space functions $\underline{f}, \underline{g}$. The failure of associativity of $\star_{\mathcal{F}}$ is due to the failure of the cocycle condition for F: We have

$$\begin{aligned}
(\underline{f} \star_{F\,F'} \underline{g}) \star_{F\,F'} \underline{h} &= \mu_{F\,F'} \circ (\mu_{F\,F'} \otimes \mathrm{id})(\underline{f} \otimes \underline{g} \otimes \underline{h}) \\
&= \mu_{F'} \circ F^{-1} \circ (\mu_{F'} \otimes 1) \circ (F^{-1} \otimes \mathrm{id})(\underline{f} \otimes \underline{g} \otimes \underline{h}) \\
&= \mu_{F'} \circ (\mu_{F'} \otimes \mathrm{id}) \circ (\Delta^{F'} \otimes \mathrm{id})F^{-1}\,(F^{-1} \otimes 1)(\underline{f} \otimes \underline{g} \otimes \underline{h}) \\
&= \mu_{F'} \circ (\mathrm{id} \otimes \mu_{F'}) \circ (\mathrm{id} \otimes \Delta^{F'})F^{-1}\,(1 \otimes F^{-1})\,\mathrm{e}^{2R}(\underline{f} \otimes \underline{g} \otimes \underline{h})
\end{aligned} \tag{4.24}$$

where in the third line we used (4.20) on the first tensor product entry of F^{-1}, while in the fourth line we used associativity of the $\star_{F'}$ product in the form $\mu_{F'} \circ (\mu_{F'} \otimes \mathrm{id}) = \mu_{F'} \circ (\mathrm{id} \otimes \mu_{F'})$ together with the inverse of the 2-cochain property of Corollary 4.18. Similarly, we have

$$\begin{aligned}
\underline{f} \star_{F\,F'} (\underline{g} \star_{F\,F'} \underline{h}) &= \mu_{F\,F'} \circ (\mathrm{id} \otimes \mu_{F\,F'})(\underline{f} \otimes \underline{g} \otimes \underline{h}) \\
&= \mu_{F'} \circ (\mathrm{id} \otimes \mu_{F'})(\mathrm{id} \otimes \Delta^{F'})F^{-1}\,(1 \otimes F^{-1})(\underline{f} \otimes \underline{g} \otimes \underline{h}) \ .
\end{aligned} \tag{4.25}$$

Comparison of these products gives (3.28).

For later use, we note that we have chosen to derive the nonassociativity structure of the $\star_{\mathcal{F}}$ product using associativity of the $\star_{F'}$ product and the 2-cochain property of Corollary 4.18. We could equally as well have decomposed the $\star_{\mathcal{F}}$ product as $\mu_{\mathcal{F}} = \mu \circ \mathcal{F}^{-1}$ and then, following similar steps as in (4.24), used associativity of the undeformed product μ and the 2-cochain property for $\mathcal{F} = F\,F'$ (or for $\mathcal{F}^{-1}$) which is not difficult to show to be given by

$$(\Delta\otimes\mathrm{id})\mathcal{F}^{-1}\,(\mathcal{F}^{-1}\otimes 1) = (\mathrm{id}\otimes\Delta)\mathcal{F}^{-1}\,(1\otimes\mathcal{F}^{-1})\ \mathrm{e}^{\,2R}\ . \tag{4.26}$$

In a similar vein, using the isomorphism $C^\infty(\mathcal{M})_{\star_F} \cong \mathsf{Diff}(M)$ of noncommutative associative algebras and the representation (4.3) of the Hopf algebra $U(\mathfrak{h})$, one can deform the algebra of differential operators on configuration space M to a quasi-associative noncommutative algebra $\mathsf{Diff}(M)_{\star_{F'}}$.

4.3 Configuration space triproducts

Let us now demonstrate how to reproduce the triproducts of Section 3 within the present formalism. The pullback of the product on the leaf of constant momentum $\bar p$ is given by

$$\begin{aligned} s_{\bar p}^*(\,\underline{f}\ \star_{F\,F'}\ \underline{g}\,) &= \mu_{F'}\big(F^{-1}(s_{\bar p}^*\underline{f}\ \otimes\ s_{\bar p}^*\underline{g}\,)\big) \\ &= \mu_{F'}\big(s_{\bar p}^*\underline{f}\ \otimes\ s_{\bar p}^*\underline{g}\,\big)\ =\ \mu\big(\exp\big(\tfrac{\mathrm{i}\,\ell_s^4}{6\hbar}\,\theta_{\bar p}\big)(s_{\bar p}^*\underline{f}\ \otimes\ s_{\bar p}^*\underline{g}\,)\big) \end{aligned} \tag{4.27}$$

where in the second equality we observed that F acts as the identity on functions of constant momentum, while in the third equality we recalled the definition of the bivector $\theta_{\bar p}$ from (3.4). In particular if f,g are functions on configuration space and $\underline{f}=\pi^*f$, $\underline{g}=\pi^*g$, then we recover the 2-product $\mu^{(2)}_{\bar p}$ from (3.6),

$$s_{\bar p}^*\big(\pi^*f\ \star_{F\,F'}\ \pi^*g\big) = f\star_{\bar p} g = \mu^{(2)}_{\bar p}(f\otimes g)\ . \tag{4.28}$$

We now proceed to the product of three phase space functions and its pullback on the leaf of constant momentum $\bar p$. We substitute in the triple product expression (4.24) the inverse of the second identity of Proposition 4.15 in order to express $\Delta^{F'}$ in terms of Δ and obtain

$$(\,\underline{f}\ \star_{F\,F'}\ \underline{g}\,)\ \star_{F\,F'}\ \underline{h} = \mu_{F'}\circ(\mu_{F'}\otimes\mathrm{id})\circ\,\mathrm{e}^{\,R}\,(\Delta\otimes\mathrm{id})F^{-1}\,(F^{-1}\otimes 1)(\,\underline{f}\ \otimes\ \underline{g}\ \otimes\ \underline{h}\,)\ . \tag{4.29}$$

The pullback of this expression reads as

$$\begin{aligned} s_{\bar p}^*\big((\,\underline{f}\ \star_{F\,F'}\ \underline{g}\,)\star_{F\,F'}\ \underline{h}\,\big) &= \mu_{F'}\circ(\mu_{F'}\otimes\mathrm{id})\circ\,\mathrm{e}^{\,R}\,(\Delta\otimes\mathrm{id})F^{-1}\,(F^{-1}\otimes 1)\big(s_{\bar p}^*\underline{f}\ \otimes s_{\bar p}^*\underline{g}\ \otimes\ s_{\bar p}^*\underline{h}\,\big) \\ &= \star_{\bar p}\big(\,\mathrm{e}^{\,R}(s_{\bar p}^*\underline{f}\ \otimes s_{\bar p}^*\underline{g}\ \otimes\ s_{\bar p}^*\underline{h}\,)\big) \end{aligned} \tag{4.30}$$

where we dropped both F^{-1} and $(\Delta\otimes\mathrm{id})F^{-1} = \exp\big(\frac12\,(\Delta(P_i)\otimes\tilde P^i - \Delta(\tilde P^i)\otimes P_i)\big)$ because they act trivially on functions of constant momentum. In particular if f,g,h are functions on configuration space and $\underline{f}=\pi^*f$, $\underline{g}=\pi^*g$, $\underline{h}=\pi^*h$, then we recover the basic triproduct $\mu^{(3)}_{\bar p}$ from (3.15), i.e.,

$$s_{\bar p}^*\big((\pi^*f\ \star_{F\,F'}\ \pi^*g\,)\star_{F\,F'}\ \pi^*h\big) = \star_{\bar p}\big(\,\mathrm{e}^{\,R}(f\otimes g\otimes h)\big) = \mu^{(3)}_{\bar p}(f\otimes g\otimes h)\ . \tag{4.31}$$

To generalise this computation to the pullback of the product of n functions we introduce the notation $F = F^\alpha \otimes F_\alpha \in U(\mathfrak{h}) \otimes U(\mathfrak{h})$ (with summation over α understood) and define $F_{12} := F \otimes 1$, $F_{23} := 1 \otimes F$, $F_{13} := F^\alpha \otimes 1 \otimes F_\alpha$ in $U(\mathfrak{h})^{\otimes 3}$. More generally $F_{ab} \in U(\mathfrak{h})^{\otimes n}$ is the element which is non-trivial only in the a-th and b-th factors of the tensor product: $F_{ab} = 1 \otimes \cdots \otimes F^\alpha \otimes \cdots \otimes F_\alpha \otimes \cdots \otimes 1$. Similarly, in $U(\mathfrak{h})^{\otimes n}$ we set $R_{123} = R \otimes 1 \otimes \cdots \otimes 1$ and more generally

$$R_{abc} = \tfrac{\ell_s^4}{12}\, R^{ijk}\, P_i^a\, P_j^b\, P_k^c \ , \tag{4.32}$$

where as before $\zeta^a \in U(\mathfrak{h})^{\otimes n}$ for $\zeta \in U(\mathfrak{h})$ is the element which is non-trivial only in the a-th factor of the tensor product: $\zeta^a = 1 \otimes \cdots \otimes \zeta \otimes \cdots \otimes 1$.

With these notations we have

$$\begin{aligned}(\Delta \otimes \mathrm{id})F^{-1} &= (\Delta \otimes \mathrm{id}) \exp\big(\tfrac12\,(P_i \otimes \tilde P^i - \tilde P^i \otimes P_i)\big) \\ &= \exp\Big(\tfrac12\,\big(\Delta(P_i) \otimes \tilde P^i - \Delta(\tilde P^i) \otimes P_i\big)\Big) \ = \ F_{13}^{-1}\, F_{23}^{-1} \ ,\end{aligned} \tag{4.33}$$

where in the last step we expanded the coproduct and then observed that the arguments of the exponential mutually commute. Substituting this equality in the second identity of Proposition 4.15 we obtain $\big(\Delta^{F'} \otimes \mathrm{id}\big)F^{-1} = \mathrm{e}^{\,R}\, F_{13}^{-1}\, F_{23}^{-1}$, which is immediately generalized to

$$\big(\Delta^{F'} \otimes \mathrm{id}^{\,\otimes n-1}\big)F_{1e}^{-1} \ = \ \mathrm{e}^{\,R_{12\,e+1}}\, F_{1\,e+1}^{-1}\, F_{2\,e+1}^{-1} \ . \tag{4.34}$$

Using as in (4.33) multiplicativity of the coproduct and commutativity of the momentum algebra we also easily obtain $\big(\Delta^{F'} \otimes \mathrm{id}^{\otimes 2}\big)\, \mathrm{e}^{\,R} = \mathrm{e}^{\,R_{134}}\, \mathrm{e}^{\,R_{234}}$ which is immediately generalized to

$$\big(\Delta^{F'} \otimes \mathrm{id}^{\,\otimes n-1}\big)\, \mathrm{e}^{\,R_{1bc}} \ = \ \mathrm{e}^{\,R_{1\,b+1\,c+1}}\ \mathrm{e}^{\,R_{2\,b+1\,c+1}} \ . \tag{4.35}$$

Proposition 4.36.

$$\begin{aligned}&\big(\cdots(\,\underline{f}_1 \star_{F\,F'} \underline{f}_2\,) \star_{F\,F'} \cdots \star_{F\,F'} \underline{f}_{n-1}\big) \star_{F\,F'} \underline{f}_n \\ &\qquad = \ \mu_{F'} \circ (\mu_{F'} \otimes \mathrm{id}) \circ \cdots \circ (\mu_{F'} \otimes \mathrm{id}^{\,\otimes n-2}) \\ &\qquad\qquad \circ \prod_{1\le a<b<c\le n} \mathrm{e}^{\,R_{abc}} \prod_{1\le d<e\le n} F_{de}^{-1}\big(\,\underline{f}_1 \otimes \cdots \otimes \underline{f}_n\big) \ .\end{aligned}$$

Proof. The proof is by induction. The assertion holds for $n=3$ by (4.29). We suppose that it holds for $n>3$ and prove that it holds for $n+1$ by computing

$$\begin{aligned}&\big(\cdots(\,\underline{f}_1 \star_{F\,F'} \underline{f}_2) \star_{F\,F'} \cdots \star_{F\,F'} \underline{f}_n\big) \star_{F\,F'} \underline{f}_{n+1} \\ &\quad = \ \mu_{F'} \circ (\mu_{F'} \otimes \mathrm{id}) \circ \cdots \circ (\mu_{F'} \otimes \mathrm{id}^{\,\otimes n-2}) \\ &\qquad \circ \prod_{1\le a<b<c\le n} \mathrm{e}^{\,R_{abc}} \prod_{1\le d<e\le n} F_{de}^{-1}\big(\cdots(\,\underline{f}_1 \star_{F\,F'} \underline{f}_2) \otimes \cdots \otimes \underline{f}_n\big) \otimes \underline{f}_{n+1} \\ &\quad = \ \mu_{F'} \circ (\mu_{F'} \otimes \mathrm{id}) \circ \cdots \circ (\mu_{F'} \otimes \mathrm{id}^{\,\otimes n-2}) \\ &\qquad \circ \prod_{1\le a<b<c\le n} \mathrm{e}^{\,R_{abc}} \prod_{1\le d<e\le n} F_{de}^{-1}\, (\mu_{F'} \otimes \mathrm{id}^{\,\otimes n-1}) F_{12}^{-1}\big(\,\underline{f}_1 \otimes \cdots \otimes \underline{f}_n \otimes \underline{f}_{n+1}\big) \\ &\quad = \ \mu_{F'} \circ (\mu_{F'} \otimes \mathrm{id}) \circ \cdots \circ (\mu_{F'} \otimes \mathrm{id}^{\,\otimes n-1}) \\ &\qquad \circ \Bigg(\big(\Delta^{F'} \otimes \mathrm{id}^{\,\otimes n-1}\big)\Big(\prod_{1\le a<b<c\le n} \mathrm{e}^{\,R_{abc}} \prod_{1\le d<e\le n} F_{de}^{-1}\Big)\Bigg)\, F_{12}^{-1}\big(\,\underline{f}_1 \otimes \cdots \otimes \underline{f}_n \otimes \underline{f}_{n+1}\big) \\ &\quad = \ \mu_{F'} \circ (\mu_{F'} \otimes \mathrm{id}) \circ \cdots \circ (\mu_{F'} \otimes \mathrm{id}^{\,\otimes n-1})\end{aligned} \tag{4.37}$$

$$\circ \prod_{1\le a<b<c\le n+1} \mathrm{e}^{\,R_{abc}} \prod_{1\le d<e\le n+1} F_{de}^{-1}\big(\,\underline{f}_1\otimes\cdots\otimes \underline{f}_n\otimes \underline{f}_{n+1}\big)$$

where in the third step we used (4.20), and in the last step we used (4.35) to write

$$\big(\Delta^{F'}\otimes \mathrm{id}^{\otimes\, n-1}\big) \prod_{1\le a<b<c\le n} \mathrm{e}^{\,R_{abc}} = \prod_{3\le b<c\le n+1} \mathrm{e}^{\,R_{1bc}} \prod_{2\le a<b<c\le n+1} \mathrm{e}^{\,R_{abc}} \tag{4.38}$$

while from (4.34) we obtain

$$\begin{aligned}\big(\Delta^{F'}\otimes \mathrm{id}^{\otimes n-1}\big) \prod_{1\le d<e\le n} F_{de}^{-1} &= \prod_{3\le e<n+1} \mathrm{e}^{\,R_{12\,e}} \prod_{3\le e\le n+1} F_{1e}^{-1} \prod_{2\le d<e\le n+1} F_{de}^{-1} \\ &= \prod_{3\le e<n+1} \mathrm{e}^{\,R_{12\,e}} \prod_{1\le d<e\le n+1} F_{de}^{-1}\, F_{12} \end{aligned}\tag{4.39}$$

and the result follows. □

Proposition 3.9 now follows as an easy corollary. In Propositon 4.36 the inverse twists F_{de}^{-1} have been ordered on the right and therefore when we pullback this result along $s_{\bar p}^*$, as in (4.30) we can drop these terms because their action is trivial. We therefore obtain

$$\begin{aligned} & s_{\bar p}^*\big[\big(\cdots\big(\,\underline{f}_1 \star_{F\,F'} \underline{f}_2\big) \star_{F\,F'} \cdots \star_{F\,F'} \underline{f}_{n-1}\,\big) \star_{F\,F'} \underline{f}_{\,n}\big] \\ &\qquad = \ \mu_{F'}\circ(\mu_{F'}\otimes \mathrm{id})\circ\cdots\circ(\mu_{F'}\otimes \mathrm{id}^{\otimes\, n-2})\circ \prod_{1\le a<b<c\le n} \mathrm{e}^{\,R_{abc}}\big(\,\underline{f}_1\otimes\cdots\otimes \underline{f}_n\big)\ , \end{aligned}\tag{4.40}$$

and Proposition 3.9 follows immediately by setting $\underline{f}_i = \pi^* f_i$.

4.4 Differential forms and tensor fields

The algebraic techniques we have described in this section can be applied to any algebra carrying a Hopf algebra symmetry. Hence we can extend our results by considering the larger algebra of exterior differential forms rather than just the algebra of functions. There is a natural extension of the representation (4.2) of the Lie algebra $\mathfrak{h}$ on $C^\infty(\mathcal{M}) = \Omega^0(\mathcal{M})$ to the vector space of exterior forms $\Omega^\bullet(\mathcal{M}) = C^\infty(\mathcal{M}, \bigwedge^\bullet T^*\mathcal{M}) = \bigoplus_{n\ge0}\Omega^n(\mathcal{M})$; it is given by the Lie derivative $\mathcal{L}$. This representation is extended to all of $U(\mathfrak{h})$ in the obvious way via $\mathcal{L}_{\xi\,\zeta} = \mathcal{L}_\xi\circ\mathcal{L}_\zeta$ for all $\xi,\zeta \in U(\mathfrak{h})$; for example $\mathcal{L}_{P_i\,P_j} = \mathcal{L}_{P_i}\circ\mathcal{L}_{P_j}$. Furthermore, $\Omega^\bullet(\mathcal{M})$ is an algebra with the associative exterior product $\wedge$, and the Hopf algebra $U(\mathfrak{h})$ is a symmetry of this algebra because of the Leibniz rule for the Lie derivative, or equivalently because the exterior product is compatible with the coproduct of elements of $U(\mathfrak{h})$ acting on forms (via the Lie derivative). For ease of notation we set $\zeta(\,\underline{\eta}\,) = \mathcal{L}_\zeta(\,\underline{\eta}\,)$, for all $\zeta\in U(\mathfrak{h})$ and $\underline{\eta}\,,\,\underline{\omega}\in\Omega^\bullet(\mathcal{M})$, so that by (4.6) the compatibility condition reads as

$$\zeta(\,\underline{\eta}\wedge\underline{\omega}\,) = \wedge\big(\Delta(\zeta)(\,\underline{\eta}\otimes\underline{\omega}\,)\big)\ . \tag{4.41}$$

The algebra of exterior forms $\Omega^\bullet(\mathcal{M})$ can then be deformed to the exterior algebra $\Omega^\bullet(\mathcal{M})_{\star_{F'}}$, as vector spaces $\Omega^\bullet(\mathcal{M}) = \Omega^\bullet(\mathcal{M})_{\star_{F'}}$; the deformed exterior product $\wedge_{\star_{F'}}$ in $\Omega^\bullet(\mathcal{M})_{\star_{F'}}$ is given by

$$\underline{\eta}\wedge_{\star_{F'}}\underline{\omega} = \wedge\big(F'^{-1}(\,\underline{\eta}\otimes\underline{\omega}\,)\big)\ , \tag{4.42}$$

for all $\underline{\eta}\,,\,\underline{\omega}\in\Omega^\bullet(\mathcal{M})_{\star_{F'}}$.

Now all expressions in Sections 4.2 and 4.3 have been obtained by solely using:

- Hopf algebra properties of the twists F and F' of Proposition 4.15 and Corollary 4.18.
- Compatibility of the action of $U(\mathfrak{h})$ on $C^\infty(\mathcal{M})$ with the coproduct in $U(\mathfrak{h})$ and the product in $C^\infty(\mathcal{M})$ (and also in (4.24)–(4.25) the associativity of the product $\mu_{F'}$ in $C^\infty(\mathcal{M})_{\star_{F'}}$).
- Triviality of the action of momentum translations $\tilde{P}^i \in U(\mathfrak{h})$ on the images of the pullbacks $s^*_{\bar{p}}$ of sections $s_{\bar{p}} : M \to \mathcal{M}$ of constant momentum $\bar{p}$; this in particular implies that the twist F acts as the identity on the image of $s^*_{\bar{p}}$.

Since the pullbacks $s^*_{\bar{p}}$ and π^* naturally extend to exterior forms, so that the three conditions above also hold true for the algebra $\Omega^\bullet(\mathcal{M})$, we conclude that all expressions in Section 4.2 and in Section 4.3 hold true also when we replace the product μ with the exterior product $\wedge$ and functions $\underline{f}_i \in C^\infty(\mathcal{M})$ or $f_i \in C^\infty(M)$ with forms $\underline{\eta}_i \in \Omega^\bullet(\mathcal{M})$ or $\eta_i \in \Omega^\bullet(M)$, and the action of the universal enveloping algebra $U(\mathfrak{h})$ on forms is always via the Lie derivative. In particular we have

$$\begin{aligned} \wedge^{(n)}_{\bar{p}} (\eta_1 \otimes \cdots \otimes \eta_n) &:= s^*_{\bar{p}}\big[\big(\cdots(\pi^*\eta_1 \wedge_{\star_{F\,F'}} \pi^*\eta_2) \wedge_{\star_{F\,F'}} \cdots\big) \wedge_{\star_{F\,F'}} \pi^*\eta_n\big] \\ &= \wedge_{\star_{\bar{p}}} \circ \prod_{1\le a<b<c\le n} \mathrm{e}^{\,R_{abc}}\,(\eta_1 \otimes \cdots \otimes \eta_n)\ , \end{aligned} \tag{4.43}$$

for all $\eta_i \in \Omega^\bullet(M)$, $i = 1, \dots, n$, where $\wedge_{\star_{\bar{p}}}$ is the Moyal–Weyl star product on forms (cf. (3.8))

$$\wedge_{\star_{\bar{p}}} (\eta_1 \otimes \cdots \otimes \eta_n) = \wedge\Big[\exp\Big(\frac{\mathrm{i}\,\ell_s^4}{6\hbar} \sum_{1\le a<b\le n} \theta^{ij}_{\bar{p}}\,\mathcal{L}^a_{\partial_i}\,\mathcal{L}^b_{\partial_j}\Big)(\eta_1 \otimes \cdots \otimes \eta_n)\Big]\ . \tag{4.44}$$

In the general context of integration of forms, we also find that the products $\wedge^{(n)}_{\bar{p}}$ under integration reduce to the Moyal–Weyl exterior product $\wedge_{\star_{\bar{p}}}$.

Proposition 4.45. *If $\eta_1, \dots, \eta_n \in \Omega^\bullet(M)$ are forms on configuration space M such that $\eta_1 \wedge \cdots \wedge \eta_n$ is a top form in $\Omega^\bullet(M)$, then*

$$\int_M \wedge^{(n)}_{\bar{p}}(\eta_1 \otimes \cdots \otimes \eta_n) = \int_M \eta_1 \wedge_{\star_{\bar{p}}} \cdots \wedge_{\star_{\bar{p}}} \eta_n\ .$$

Proof. We first compute

$$\begin{aligned} \wedge^{(n)}_{\bar{p}} (\eta_1 \otimes \ldots \otimes \eta_n) &= \wedge_{\star_{\bar{p}}} \circ \exp\Big(\sum_{1\le a<b<c\le n} R_{abc}\Big)(\eta_1 \otimes \cdots \eta_n) \\ &= \eta_1 \wedge_{\star_{\bar{p}}} \cdots \wedge_{\star_{\bar{p}}} \eta_n + \wedge_{\star_{\bar{p}}} \circ \sum_{1\le a<b<c\le n} R_{abc} \circ \mathcal{O}\,(\eta_1 \otimes \cdots \otimes \eta_n) \\ &= \eta_1 \wedge_{\star_{\bar{p}}} \cdots \wedge_{\star_{\bar{p}}} \eta_n \\ &\quad + \frac{\ell_s^4}{12} \wedge_{\star_{\bar{p}}} \circ R^{ijk} \sum_{1\le a<b<c\le n} \mathcal{L}^a_{\partial_i}\,\mathcal{L}^b_{\partial_j}\,\mathcal{L}^c_{\partial_k} \circ \mathcal{O}\,(\eta_1 \otimes \cdots \otimes \eta_n) \\ &= \eta_1 \wedge_{\star_{\bar{p}}} \cdots \wedge_{\star_{\bar{p}}} \eta_n \\ &\quad + \frac{\ell_s^4}{12} \wedge_{\star_{\bar{p}}} \circ R^{ijk} \sum_{b=2}^{n-1} \mathcal{L}^b_{\partial_j} \sum_{a=1}^{b-1} \mathcal{L}^a_{\partial_i} \sum_{c=b+1}^{n} \mathcal{L}^c_{\partial_k} \circ \mathcal{O}\,(\eta_1 \otimes \cdots \otimes \eta_n) \\ &= \eta_1 \wedge_{\star_{\bar{p}}} \cdots \wedge_{\star_{\bar{p}}} \eta_n \end{aligned} \tag{4.46}$$

$$
\begin{aligned}
&\quad + \frac{\ell_s^4}{12} \wedge_{\star_{\bar p}} \circ R^{ijk} \sum_{b=2}^{n-1} \sum_{e=1}^{n} \mathcal{L}^e_{\partial_j} \sum_{a=1}^{b-1} \mathcal{L}^a_{\partial_i} \sum_{c=b+1}^{n} \mathcal{L}^c_{\partial_k} \circ \mathcal{O}\,(\eta_1 \otimes \cdots \otimes \eta_n) \\
&= \eta_1 \wedge_{\star_{\bar p}} \cdots \wedge_{\star_{\bar p}} \eta_n \\
&\quad + \frac{\ell_s^4}{12} \mathcal{L}_{\partial_j} \circ \wedge_{\star_{\bar p}} \circ R^{ijk} \sum_{b=2}^{n-1} \sum_{a=1}^{b-1} \mathcal{L}^a_{\partial_i} \sum_{c=b+1}^{n} \mathcal{L}^c_{\partial_k} \circ \mathcal{O}\,(\eta_1 \otimes \cdots \otimes \eta_n) \ .
\end{aligned}
$$

In the second line we expanded the exponential by factoring the operator $\sum_{a<b<c} R_{abc}$ as

$$
\begin{aligned}
\exp\Big(\sum_{1\le a<b<c\le n} R_{abc} \Big) &= \mathrm{id} + \sum_{1\le a<b<c\le n} R_{abc} + \frac{1}{2}\Big(\sum_{1\le a<b<c\le n} R_{abc} \Big)^2 + \cdots \\
&=: \mathrm{id} + \sum_{1\le a<b<c\le n} R_{abc} \circ \mathcal{O} \ . \qquad (4.47)
\end{aligned}
$$

In the fifth line we used antisymmetry of R^{ijk} to replace $\mathcal{L}^b_{\partial_j}$ with $\sum_{e=1}^n \mathcal{L}^e_{\partial_j}$. In the last line we used the Leibniz rule

$$
\mathcal{L}_{\partial_i}(\eta_1 \wedge_{\star_{\bar p}} \cdots \wedge_{\star_{\bar p}} \eta_n) = \wedge_{\star_{\bar p}} \circ \sum_{e=1}^{n} \mathcal{L}^e_{\partial_j}(\eta_1 \otimes \cdots \otimes \eta_n) \ . \qquad (4.48)
$$

Next we use the Cartan formula for the Lie derivative in terms of the exterior derivative and the contraction operator as $\mathcal{L}_{\partial_j} = \imath_{\partial_j} \circ \mathrm{d} + \mathrm{d} \circ \imath_{\partial_j}$, observe that when acting on a top form it simplifies to $\mathcal{L}_{\partial_j} = \mathrm{d} \circ \imath_{\partial_j}$, and the result then follows by integrating (4.46). □

Similarly to the exterior algebra one can also deform the tensor algebra. As for the deformed exterior product $\wedge_{\star_{\mathcal{F}}}$, the deformed tensor product $\otimes_{C^\infty(\mathcal{M})_{\star_{\mathcal{F}}}}$ is defined by composing the usual tensor product $\otimes_{C^\infty(\mathcal{M})}$ over $C^\infty(\mathcal{M})$ with the inverse twist: $\otimes_{C^\infty(\mathcal{M})_{\star_{\mathcal{F}}}} := \otimes_{C^\infty(\mathcal{M})} \circ \mathcal{F}^{-1}$.

4.5 Phase space diffeomorphisms

The Drinfeld twist deformation procedure we have been implementing consists in deforming algebras that carry a compatible representation of the Hopf algebra $U(\mathfrak{h})$ (cf. the first two items in the list of Section 4.4). We recall that the representation is compatible with the product in the algebra if the action of Lie algebra elements $X \in \mathfrak{h}$ on products is given by the Leibniz rule (and hence for elements $\zeta \in U(\mathfrak{h})$ by the coproduct). We shall now apply this procedure to the Lie algebra of vector fields $\mathsf{Vect}(\mathcal{M}) = C^\infty(\mathcal{M}, T\mathcal{M})$ on phase space, i.e., the Lie algebra of infinitesimal (local) diffeomorphisms. This is a nonassociative algebra with product given by the Lie bracket $[\ \ ] : \mathsf{Vect}(\mathcal{M}) \otimes \mathsf{Vect}(\mathcal{M}) \to \mathsf{Vect}(\mathcal{M})$. Thanks to the representation (4.2) the Lie algebra $\mathfrak{h}$ can be regarded as a subalgebra of $\mathsf{Vect}(\mathcal{M})$ and therefore its action is given by the Lie derivative: $\mathcal{L}_X(\,\underline{u}\,) = [X, \underline{u}\,]$ for all $X \in \mathfrak{h}$, $\underline{u} \in \mathsf{Vect}(\mathcal{M})$. The compatibility condition is satisfied because the Jacobi identity $[X, [\,\underline{u}\,,\,\underline{v}\,]] = [[X, \underline{u}\,], \underline{v}\,] + [\,\underline{u}\,,\,[X, \underline{v}\,]]$ is the Leibniz rule with respect to the product $[\ \ ]$.

We can thus apply the Drinfeld twist deformation procedure with 2-cochain $\mathcal{F} = F\,F'$ and obtain the deformed algebra of vector fields $\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal{F}}} = \mathsf{Vect}(\mathcal{M})_{\star_{F\,F'}}$, which as a vector space is the same as $\mathsf{Vect}(\mathcal{M})$ but has a deformed Lie bracket

$$
[\ \ ]_{\star_{\mathcal{F}}} \ : \ \mathsf{Vect}(\mathcal{M})_{\star_{\mathcal{F}}} \otimes \mathsf{Vect}(\mathcal{M})_{\star_{\mathcal{F}}} \ \longrightarrow \ \mathsf{Vect}(\mathcal{M})_{\star_{\mathcal{F}}}
$$

$$\underline{u}\otimes\underline{v} \;\longmapsto\; [\,\underline{u}\,,\,\underline{v}\,]_{\star_{\mathcal F}} := [\quad]\circ(F\,F')^{-1}(\,\underline{u}\otimes\underline{v}\,)\ . \tag{4.49}$$

This can be realized as a deformed commutator. For this, we introduce the notation $\mathcal{F}=\mathcal{F}^\alpha\otimes\mathcal{F}_\alpha$, $\mathcal{F}^{-1}=\overline{\mathcal{F}}^\alpha\otimes\overline{\mathcal{F}}_\alpha$; define the *universal* $\mathcal{R}$*-matrix* $\mathcal{R}=\mathcal{F}_{21}\,\mathcal{F}^{-1}$ where $\mathcal{F}_{21}=\mathcal{F}_\alpha\otimes\mathcal{F}^\alpha$ and denote its inverse by $\mathcal{R}^{-1}=\overline{\mathcal{R}}^\alpha\otimes\overline{\mathcal{R}}_\alpha$. We then compute

$$\begin{aligned}
[\,\underline{u}\,,\,\underline{v}\,]_{\star_{\mathcal F}} &= \big[\,\overline{\mathcal{F}}^\alpha(\,\underline{u}\,),\overline{\mathcal{F}}_\alpha(\,\underline{v}\,)\big] \\
&= \overline{\mathcal{F}}^\alpha(\,\underline{u}\,)\,\overline{\mathcal{F}}_\alpha(\,\underline{v}\,)-\overline{\mathcal{F}}_\alpha(\,\underline{v}\,)\,\overline{\mathcal{F}}^\alpha(\,\underline{u}\,) \;=\; \underline{u}\star_{\mathcal F}\underline{v}-\overline{\mathcal{R}}^\alpha(\,\underline{v}\,)\star_{\mathcal F}\overline{\mathcal{R}}_\alpha(\,\underline{u}\,)\ ,
\end{aligned} \tag{4.50}$$

where we wrote the undeformed Lie bracket as a commutator and introduced the $\star_{\mathcal F}$ product between vector fields $\underline{u}\star_{\mathcal F}\underline{v}:=\overline{\mathcal{F}}^\alpha(\,\underline{u}\,)\,\overline{\mathcal{F}}_\alpha(\,\underline{v}\,)$ which is a deformation of the product on the universal enveloping algebra of $\mathsf{Vect}(\mathcal{M})$. It is easy to see that the bracket $[\quad]_{\star_{\mathcal F}}$ has the $\star_{\mathcal F}$ antisymmetry property

$$[\,\underline{u}\,,\,\underline{v}\,]_{\star_{\mathcal F}}=\big[\,\overline{\mathcal{F}}^\alpha(\,\underline{u}\,),\overline{\mathcal{F}}_\alpha(\,\underline{v}\,)\big]=-\big[\,\overline{\mathcal{F}}_\alpha(\,\underline{v}\,),\overline{\mathcal{F}}^\alpha(\,\underline{u}\,)\big]=-\big[\,\overline{\mathcal{R}}^\alpha(\,\underline{v}\,),\overline{\mathcal{R}}_\alpha(\,\underline{u}\,)\big]_{\star_{\mathcal F}}\ . \tag{4.51}$$

We can write this relation as $[\quad]\circ\mathcal{F}^{-1}(\,\underline{u}\otimes\underline{v}\,)=-[\quad]\circ\mathcal{F}^{-1}\,\mathcal{R}^{-1}\circ\sigma(\,\underline{u}\otimes\underline{v}\,)$ where σ is the transposition $\sigma(\,\underline{u}\otimes\underline{v}\,)=\underline{v}\otimes\underline{u}$. This notation emphasizes the antisymmetry of the $\star_{\mathcal F}$ bracket with respect to transpositions implemented by the operator $\mathcal{R}^{-1}\circ\sigma$.

We can next implement the transposition of the elements $\underline{u}$ and $\underline{v}$ in the triple tensor product $\underline{u}\otimes(\underline{v}\otimes\underline{z})\in\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal F}}\otimes(\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal F}}\otimes\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal F}})$. The bracketing hints that we are later on going to apply the operator $\mathrm{id}\otimes[\quad]$. There is actually deeper information in the bracketing [16, 7], as it defines the action of the $\mathcal{F}$ deformed Hopf algebra of translations and Bopp shifts on triple tensor products. We therefore first have to identify $\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal F}}\otimes(\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal F}}\otimes\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal F}})$ with $(\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal F}}\otimes\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal F}})\otimes\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal F}}$ as quasi-Hopf algebra representations, and this is done via the action of the inverse associator $\Phi^{-1}=\mathrm{e}^{\,-2R}$; then we can apply the operator $\mathcal{R}_{12}^{-1}\circ\sigma_{12}=(\mathcal{R}^{-1}\circ\sigma)\otimes\mathrm{id}$ and finally go back to $\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal F}}\otimes(\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal F}}\otimes\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal F}})$ via $\mathrm{e}^{\,2R}$. Thus the transposition map in this case is $\mathrm{e}^{\,2R}\circ\mathcal{R}_{12}^{-1}\circ\sigma_{12}\circ\mathrm{e}^{\,-2R}$ which, since $\mathrm{e}^{\,R}$ is central and $R_{123}=-R_{213}$, simplifies to $\mathcal{R}_{12}^{-1}\circ\sigma_{12}\circ\mathrm{e}^{\,-4R}$. Hence by setting (with summation over the indices $^{\bar\phi\bar\phi}$ understood)

$$\underline{u}^{\bar\phi\bar\phi}\otimes\underline{v}^{\bar\phi\bar\phi}\otimes\underline{z}^{\bar\phi\bar\phi}:=\mathrm{e}^{\,-4R}(\,\underline{u}\otimes\underline{v}\otimes\underline{z}\,)\ , \tag{4.52}$$

it explicitly acts as

$$\begin{aligned}
\mathrm{e}^{\,2R}\circ\mathcal{R}_{12}^{-1}\circ\sigma_{12}\circ\mathrm{e}^{\,-2R}(\,\underline{u}\otimes\underline{v}\otimes\underline{z}\,) &= \mathcal{R}_{12}^{-1}\circ\sigma_{12}\circ\mathrm{e}^{\,-4R}(\,\underline{u}\otimes\underline{v}\otimes\underline{z}\,) \\
&= \overline{\mathcal{R}}^\alpha(\,\underline{v}^{\bar\phi\bar\phi}\,)\otimes\overline{\mathcal{R}}_\alpha(\,\underline{u}^{\bar\phi\bar\phi}\,)\otimes\underline{z}^{\bar\phi\bar\phi}\ .
\end{aligned} \tag{4.53}$$

Independently of its quasi-Hopf algebraic aspects the relevance of the expression (4.53) is in its appearence in the $\star_{\mathcal F}$ Jacobi identity.

Proposition 4.54. *The deformed Lie algebra of infinitesimal diffeomorphisms is characterized by the* $\star_{\mathcal F}$ *Jacobi identity*

$$\big[\,\underline{u}\,,\,[\,\underline{v}\,,\underline{z}\,]_{\star_{\mathcal F}}\big]_{\star_{\mathcal F}}=\big[[\,\underline{u}^{\bar\phi},\underline{v}^{\bar\phi}\,]_{\star_{\mathcal F}}\,,\,\underline{z}^{\bar\phi}\,\big]_{\star_{\mathcal F}}+\big[\,\overline{\mathcal{R}}^\alpha(\,\underline{v}^{\bar\phi\bar\phi}\,)\,,\,\big[\,\overline{\mathcal{R}}_\alpha(\,\underline{u}^{\bar\phi\bar\phi}\,),\underline{z}^{\bar\phi\bar\phi}\,\big]_{\star_{\mathcal F}}\big]_{\star_{\mathcal F}}\ ,$$

for all $\underline{u}\,,\,\underline{v}\,,\,\underline{z}\in\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal F}}$*, where*

$$\underline{u}^{\bar\phi}\otimes\underline{v}^{\bar\phi}\otimes\underline{z}^{\bar\phi}:=\Phi^{-1}(\,\underline{u}\otimes\underline{v}\otimes\underline{z}\,)=\mathrm{e}^{\,-2R}(\,\underline{u}\otimes\underline{v}\otimes\underline{z}\,)$$

and $\underline{u}^{\bar\phi\bar\phi}\otimes\underline{v}^{\bar\phi\bar\phi}\otimes\underline{z}^{\bar\phi\bar\phi}$ *is defined in (4.52).*

Proof. We compute

$$
\begin{aligned}
\big[\,\underline{u}\,,\,[\,\underline{v}\,,\underline{z}\,]_{\star_{\mathcal{F}}}\,\big]_{\star_{\mathcal{F}}} &= [\quad]\circ\mathcal{F}^{-1}\circ(\mathrm{id}\otimes[\quad])\circ(1\otimes\mathcal{F}^{-1})(\,\underline{u}\otimes\underline{v}\otimes\underline{z}\,)\\
&= [\quad]\circ(\mathrm{id}\otimes[\quad])\circ(\mathrm{id}\otimes\Delta)\mathcal{F}^{-1}\,(1\otimes\mathcal{F}^{-1})(\,\underline{u}\otimes\underline{v}\otimes\underline{z}\,)
\end{aligned}\tag{4.55}
$$

and

$$
\begin{aligned}
\big[[\,\underline{u}^{\bar\phi},\underline{v}^{\bar\phi}\,]_{\star_{\mathcal{F}}}\,,\,\underline{z}^{\bar\phi}\,\big]_{\star_{\mathcal{F}}} &= [\quad]\circ([\quad]\otimes\mathrm{id})\circ(\Delta\otimes\mathrm{id})\mathcal{F}^{-1}\,(\mathcal{F}^{-1}\otimes 1)(\,\underline{u}^{\bar\phi}\otimes\underline{v}^{\bar\phi}\otimes\underline{z}^{\bar\phi}\,)\\
&= [\quad]\circ([\quad]\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta)\mathcal{F}^{-1}\,(1\otimes\mathcal{F}^{-1})(\,\underline{u}\otimes\underline{v}\otimes\underline{z}\,)
\end{aligned}\tag{4.56}
$$

where we used the 2-cochain property (4.26). It follows that

$$
\begin{aligned}
&\big[\,\overline{\mathcal{R}}^{\alpha}(\,\underline{v}^{\bar\phi\bar\phi}\,)\,,\,[\,\overline{\mathcal{R}}_{\alpha}(\,\underline{u}^{\bar\phi\bar\phi}\,)\,,\underline{z}^{\bar\phi\bar\phi}\,]_{\star_{\mathcal{F}}}\,\big]_{\star_{\mathcal{F}}}\\
&\qquad= [\quad]\circ(\mathrm{id}\otimes[\quad])\circ(\mathrm{id}\otimes\Delta)\mathcal{F}^{-1}\,(1\otimes\mathcal{F}^{-1})\,\mathcal{R}_{12}^{-1}\circ\sigma_{12}\circ\,\mathrm{e}^{\,-4R}(\,\underline{u}\otimes\underline{v}\otimes\underline{z}\,)\\
&\qquad= [\quad]\circ(\mathrm{id}\otimes[\quad])\circ(\Delta\otimes\mathrm{id})\mathcal{F}^{-1}\,(\mathcal{F}\otimes 1)\,\mathrm{e}^{\,-2R}\circ\sigma_{12}\circ\,\mathrm{e}^{\,-4R}(\,\underline{u}\otimes\underline{v}\otimes\underline{z}\,)\\
&\qquad= [\quad]\circ(\mathrm{id}\otimes[\quad])\circ\sigma_{12}\circ(\Delta\otimes\mathrm{id})\mathcal{F}^{-1}\,(\mathcal{F}^{-1}\otimes 1)\,\mathrm{e}^{\,-2R}(\,\underline{u}\otimes\underline{v}\otimes\underline{z}\,)\\
&\qquad= [\quad]\circ(\mathrm{id}\otimes[\quad])\circ\sigma_{12}\circ(\mathrm{id}\otimes\Delta)\mathcal{F}^{-1}\,(1\otimes\mathcal{F}^{-1})(\,\underline{u}\otimes\underline{v}\otimes\underline{z}\,)
\end{aligned}\tag{4.57}
$$

where in the third line we used the 2-cochain property (4.26) and noticed that since in our case $\mathcal{F}_{21}=\mathcal{F}^{-1}$ we have $\mathcal{R}^{-1}=\mathcal{F}\,\mathcal{F}_{21}^{-1}=\mathcal{F}^{2}$. In the fourth line we moved the permutation σ_{12} to the left and used $\mathrm{e}^{\,-2R}\circ\sigma_{12}=\sigma_{12}\circ\,\mathrm{e}^{\,2R}$, $\mathcal{F}\circ\sigma=\sigma\circ\mathcal{F}^{-1}$ and the fact that the undeformed coproduct $\Delta(\zeta)$ of any element $\zeta\in U(\mathfrak{h})$ is *cocommutative*, i.e., it is symmetric with respect to exchange of its two legs, see (4.4). In the last line we again used (4.26). The proof now follows by observing that the undeformed Jacobi identity is equivalent to the equality

$$
[\quad]\circ(\mathrm{id}\otimes[\quad])=[\quad]\circ([\quad]\otimes\mathrm{id})+[\quad]\circ(\mathrm{id}\otimes[\quad])\circ\sigma_{12}\tag{4.58}
$$

of operators on tensor products of vector fields. □

Note that the underlying mathematical structure we are constructing is that of a quasi-Lie algebra of a quasi-Hopf algebra. The original Hopf algebra is the universal enveloping algebra of vector fields $U(\mathsf{Vect}(\mathcal{M}))$. This is deformed to a quasi-Hopf algebra $U(\mathsf{Vect}(\mathcal{M}))^{\mathcal{F}}$ via the 2-cochain $\mathcal{F}\in U(\mathfrak{h})\otimes U(\mathfrak{h})\subset U(\mathsf{Vect}(\mathcal{M}))\otimes U(\mathsf{Vect}(\mathcal{M}))$ (with the inclusion via the representation (4.2)). The quasi-Lie algebra of vector fields $\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal{F}}}$ is the linear space $\mathsf{Vect}(\mathcal{M})$ with the bracket $[\quad]_{\star_{\mathcal{F}}}:\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal{F}}}\otimes\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal{F}}}\to\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal{F}}}$.

In the commutative case the commutator $[\,\underline{u}\,,\underline{v}\,]$ is equal to the Lie derivative $\mathcal{L}_{\underline{u}}(\,\underline{v}\,)$. This motivates the definition of the $\star_{\mathcal{F}}$ Lie derivative as

$$
\mathcal{L}^{\star_{\mathcal{F}}}_{\underline{u}}:=\mathcal{L}_{\overline{\mathcal{F}}^{\alpha}(\,\underline{u}\,)}\circ\overline{\mathcal{F}}_{\alpha}\ ,\tag{4.59}
$$

for all $\underline{u}\,,\,\underline{v}\in\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal{F}}}$, so that $\mathcal{L}^{\star_{\mathcal{F}}}_{\underline{u}}(\,\underline{v}\,)=[\,\underline{u}\,,\underline{v}\,]_{\star_{\mathcal{F}}}$. The $\star_{\mathcal{F}}$ Lie derivative is given by composing the usual Lie derivative with the inverse twist $\mathcal{F}^{-1}$. The definition (4.59) holds more generally when the $\star_{\mathcal{F}}$ Lie derivative acts on exterior forms or tensor fields. The same algebra as that of Proposition 4.54 shows that the $\star_{\mathcal{F}}$ Lie derivative satisfies the deformed Leibniz rule

$$
\mathcal{L}^{\star_{\mathcal{F}}}_{\underline{u}}(\,\underline{\eta}\wedge_{\star_{\mathcal{F}}}\underline{\omega}\,)=\mathcal{L}^{\star_{\mathcal{F}}}_{\underline{u}^{\bar\phi}}(\,\underline{\eta}^{\bar\phi}\,)\wedge_{\star_{\mathcal{F}}}\underline{\omega}^{\bar\phi}+\overline{\mathcal{R}}^{\alpha}(\,\underline{\eta}^{\bar\phi\bar\phi}\,)\wedge_{\star_{\mathcal{F}}}\mathcal{L}^{\star_{\mathcal{F}}}_{\overline{\mathcal{R}}_{\alpha}(\,\underline{u}^{\bar\phi\bar\phi}\,)}(\,\underline{\omega}^{\bar\phi\bar\phi}\,)\ ,\tag{4.60}
$$

for all $\underline{u} \in \mathsf{Vect}(\mathcal{M})_{\star_{\mathcal{F}}}$, $\underline{\eta}\,,\,\underline{\omega} \in \Omega^\bullet(\mathcal{M})_{\star_{\mathcal{F}}}$, where $\underline{u}^{\bar\phi} \otimes \underline{\eta}^{\bar\phi} \otimes \underline{\omega}^{\bar\phi} = \mathrm{e}^{-2R}(\,\underline{u} \otimes \underline{\eta} \otimes \underline{\omega}\,)$ and $\underline{u}^{\bar\phi\bar\phi} \otimes \underline{\eta}^{\bar\phi\bar\phi} \otimes \underline{\omega}^{\bar\phi\bar\phi} = \mathrm{e}^{-4R}(\,\underline{u} \otimes \underline{\eta} \otimes \underline{\omega}\,)$. For this, we note that the Leibniz rule for the undeformed Lie derivative can be written as $\mathcal{L} \circ (\mathrm{id} \otimes \wedge) = \wedge \circ (\mathcal{L} \otimes \mathrm{id}) + \wedge \circ (\mathrm{id} \otimes \mathcal{L}) \circ \sigma_{12}$, where we used the notation $\mathcal{L}(\,\underline{u}\,,\,\underline{\eta}\,) := \mathcal{L}_{\underline{u}}(\,\underline{\eta}\,)$. The Leibniz rule for tensor fields is then obtained by replacing differential forms with tensor fields and the deformed exterior product $\wedge_{\star_{\mathcal{F}}}$ with the deformed tensor product $\otimes_{C^\infty(\mathcal{M})_{\star_{\mathcal{F}}}}$.

4.6 Configuration space diffeomorphisms

We have described the deformed Lie algebra of infinitesimal diffeomorphisms on noncommutative and nonassociative phase space, and their action on phase space forms and tensors. We next study the induced deformed Lie algebra and action in configuration space. For this, we take advantage of the global coordinate system $\{x^i, p_i\}$ on phase space to define pullbacks of vector fields (more generally we could consider any phase space $\mathcal{M}$ that is foliated with leaves isomorphic to configuration space M). Any vector field on $\mathcal{M}$ is of the form $\underline{v} = v^i(x,p)\,\partial_i + \tilde v_i(x,p)\,\tilde\partial^i$. We define the Lie subalgebra of vector fields

$$ {}^H\mathsf{Vect}(\mathcal{M}) = \big\{ \underline{v} \in \mathsf{Vect}(\mathcal{M}) \ \big| \ \underline{v} = v^i(x,p)\,\partial_i \big\} \tag{4.61} $$

and observe that the Lie algebra $\mathfrak{h}$ leaves ${}^H\mathsf{Vect}(\mathcal{M})$ invariant: $\big[\mathfrak{h}, {}^H\mathsf{Vect}(\mathcal{M})\big] \subset {}^H\mathsf{Vect}(\mathcal{M})$, hence we can apply the deformation procedure to ${}^H\mathsf{Vect}(\mathcal{M})$ and obtain the deformed Lie algebra ${}^H\mathsf{Vect}(\mathcal{M})_{\star_{\mathcal{F}}}$.

There is an obvious one-to-one correspondence between coordinate vector fields ∂_i on configuration space M and coordinate vector fields ∂_i on phase space $\mathcal{M}$. Thus, on one hand we can now inject the Lie algebra $\mathsf{Vect}(M)$ in ${}^H\mathsf{Vect}(\mathcal{M})$ by defining

$$ \pi^* : \mathsf{Vect}(M) \longrightarrow {}^H\mathsf{Vect}(\mathcal{M}) \qquad \text{with} \quad v^i\,\partial_i \longmapsto \pi^*(v^i)\,\partial_i \ . \tag{4.62} $$

On the other hand, given the section $s_{\bar p} : M \to \mathcal{M}\ ,\ x \mapsto (x, \bar p)$ we can project the Lie algebra ${}^H\mathsf{Vect}(\mathcal{M})$ to $\mathsf{Vect}(M)$ by defining

$$ s_{\bar p}^* : {}^H\mathsf{Vect}(\mathcal{M}) \longrightarrow \mathsf{Vect}(M) \qquad \text{with} \quad \underline{v} = v^i\,\partial_i \longmapsto s_{\bar p}^*(v^i)\,\partial_i \ . \tag{4.63} $$

We can then characterize the deformed Lie algebra of infinitesimal diffeomorphisms $\mathsf{Vect}(M)_{\bar p}$ as the vector space $\mathsf{Vect}(M)$ with the brackets

$$ [u,v]^{(2)}_{\bar p} \ := \ s^*_{\bar p}\big([\pi^*u, \pi^*v]_{\star_{\mathcal{F}}}\big) \ , \tag{4.64} $$

$$ \big[u\,,\,[v,z]\big]^{(3)}_{\bar p} \ := \ s^*_{\bar p}\big(\big[\pi^*u\,,\,[\pi^*v, \pi^*z]_{\star_{\mathcal{F}}}\big]_{\star_{\mathcal{F}}}\big) \ , \tag{4.65} $$

$$ \big[[u,v]\,,\,z\big]^{(3)}_{\bar p} \ := \ s^*_{\bar p}\big(\big[[\pi^*u, \pi^*v]_{\star_{\mathcal{F}}}\,,\,\pi^*z\big]_{\star_{\mathcal{F}}}\big) \ . \tag{4.66} $$

Since no derivatives in the momentum directions appear, for two vector fields we find explicitly

$$ [u,v]^{(2)}_{\bar p} = s^*_{\bar p} \circ [\quad] \circ F'^{-1}\,F^{-1}(\pi^*u \otimes \pi^*v) = s^*_{\bar p} \circ [\quad] \circ F'^{-1}\big(\pi^*u \otimes \pi^*v\big) = [u,v]_{\star_{\bar p}} \tag{4.67} $$

where the bracket

$$ [u,v]_{\star_{\bar p}} := [\quad] \circ \exp\big(\tfrac{\mathrm{i}\,\ell_s^4}{6\hbar}\,\theta^{ij}_{\bar p}\,\mathcal{L}_{\partial_i} \otimes \mathcal{L}_{\partial_j}\big)(u \otimes v) \tag{4.68} $$

is the quantum Lie algebra bracket on Moyal–Weyl noncommutative space [4] (see [5, Section 8.2.3] for an elementary introduction). In terms of the twist $F_{\bar p}$ and its universal $\mathcal{R}$-matrix defined by

$$F_{\bar p} = \exp\big(-\tfrac{\mathrm{i}\,\ell_s^4}{6\hbar}\,\theta^{ij}_{\bar p}\,\mathcal{L}_{\partial_i}\otimes\mathcal{L}_{\partial_j}\big) \qquad \text{and} \qquad F_{\bar p}^{-1} = \overline{F}{}^{\alpha}_{\bar p}\otimes\overline{F}_{\bar p\,\alpha} = \exp\big(\tfrac{\mathrm{i}\,\ell_s^4}{6\hbar}\,\theta^{ij}_{\bar p}\,\mathcal{L}_{\partial_i}\otimes\mathcal{L}_{\partial_j}\big)\ ,$$
$$R_{\bar p} = F_{\bar p\,21}\,F_{\bar p}^{-1} = F_{\bar p}^{-2} \qquad \text{and} \qquad R_{\bar p}^{-1} = \overline{R}{}^{\alpha}_{\bar p}\otimes\overline{R}_{\bar p\,\alpha} = F_{\bar p}^{2}\ , \tag{4.69}$$

we have $[\ \ ]_{\star_{\bar p}} = [\ \ ]\circ F_{\bar p}^{-1}$ and the $\star_{\bar p}$ antisymmetry property

$$[u,v]_{\star_{\bar p}} = -\big[\,\overline{R}{}^{\alpha}_{\bar p}(v),\overline{R}_{\bar p\,\alpha}(u)\,\big]_{\star_{\bar p}}\ . \tag{4.70}$$

The bracket (4.67) also defines the Lie derivative on vector fields.

The explicit expression for the 3-bracket $[[u,v],z]^{(3)}_{\bar p}$ can be obtained by following the same steps as in (4.29)–(4.31). We just substitute $\mu_{F'}$ with $[\ \ ]_{F'} := [\ \ ]\circ F'^{-1}$ and functions with vector fields; we proceed similarly with the three bracket $[u,[v,z]]^{(3)}_{\bar p}$ (cf. (4.25)) and obtain

$$\begin{aligned}\big[[u,v]\,,\,z\big]^{(3)}_{\bar p} &= [\ \ ]_{\star_{\bar p}}\circ\big([\ \ ]_{\star_{\bar p}}\otimes\mathrm{id}\big)\circ\,\mathrm{e}^{\,R}(u\otimes v\otimes z)\ ,\\ \big[u\,,\,[v,z]\big]^{(3)}_{\bar p} &= [\ \ ]_{\star_{\bar p}}\circ\big(\mathrm{id}\otimes[\ \ ]_{\star_{\bar p}}\big)\circ\,\mathrm{e}^{\,-R}(u\otimes v\otimes z)\ .\end{aligned} \tag{4.71}$$

The relation between the 3-brackets $[\ \ [\ \ ]]^{(3)}_{\bar p}$ and $[[\ \ ]\ \ ]^{(3)}_{\bar p}$ can be obtained from these explicit expressions: From the $\star_{\bar p}$ antisymmetry property of the bracket $[\ \ ]_{\star_{\bar p}}$, the easily checked property of the universal $\mathcal{R}$-matrix $\big(\mathrm{id}\otimes\Delta^{F_{\bar p}}\big)R_{\bar p}^{-1} = R_{\bar p\,12}^{-1}\,R_{\bar p\,13}^{-1}$ and the identity $\zeta\circ[\ \ ]_{\star_{\bar p}} = [\ \ ]_{\star_{\bar p}}\circ\Delta^{F_{\bar p}}(\zeta)$ (cf. (4.20)), we have

$$\big[[u,v]_{\star_{\bar p}}\,,\,z\big]_{\star_{\bar p}} = -\big[\,\overline{R}{}^{\alpha}_{\bar p}(z)\,,\,\overline{R}_{\bar p\,\alpha}\big([u,v]_{\star_{\bar p}}\big)\,\big]_{\star_{\bar p}} = -\big[\,\overline{R}{}^{\beta}_{\bar p}\,\overline{R}{}^{\alpha}_{\bar p}(z)\,,\,\big[\,\overline{R}_{\bar p\,\beta}(u),\overline{R}_{\bar p\,\alpha}(v)\,\big]_{\star_{\bar p}}\,\big]_{\star_{\bar p}}\ , \tag{4.72}$$

which implies

$$\big[[u^{\bar\phi},v^{\bar\phi}]\,,\,z^{\bar\phi}\big]^{(3)}_{\bar p} = -\big[\,\overline{R}{}^{\beta}_{\bar p}\,\overline{R}{}^{\alpha}_{\bar p}(z)\,,\,\big[\,\overline{R}_{\bar p\,\beta}(u),\overline{R}_{\bar p\,\alpha}(v)\,\big]\,\big]^{(3)}_{\bar p} \tag{4.73}$$

where we used the notation

$$u^{\bar\phi}\otimes v^{\bar\phi}\otimes z^{\bar\phi} := \mathrm{e}^{\,-2R}(u\otimes v\otimes z)\ . \tag{4.74}$$

We can now induce the deformed Jacobi identity for the 3-bracket $[\ \ [\ \ ]]^{(3)}_{\bar p}$ from the phase space $\star_{\mathcal{F}}$ Jacobi identity of Proposition 4.54. We let $(\,\underline{u}\,,\underline{v}\,,\underline{z}\,) = (\pi^*u,\pi^*v,\pi^*z)$ and then pullback along $s^*_{\bar p}$. Moving F to the right of F', so that its action becomes trivial on the image of π^*, the last term simplifies as

$$\begin{aligned}&s^*_{\bar p}\big(\big[\,\overline{\mathcal{R}}{}^{\alpha}(\pi^*v^{\bar\phi\bar\phi})\,,\,\big[\,\overline{\mathcal{R}}_{\alpha}(\pi^*u^{\bar\phi\bar\phi}),\pi^*z^{\bar\phi\bar\phi}\,\big]_{\star_{\mathcal{F}}}\,\big]_{\star_{\mathcal{F}}}\big)\\ &= s^*_{\bar p}\big([\ \ ]_{F'}\circ(\mathrm{id}\otimes[\ \ ]_{F'})\circ(\mathrm{id}\otimes\Delta^{F'})F^{-1}\,(1\otimes F^{-1})\,(F'^{\,2}\otimes 1)\ \mathrm{e}^{\,4R}(\pi^*v\otimes\pi^*u\otimes\pi^*z)\big)\\ &= s^*_{\bar p}\big([\ \ ]_{F'}\circ(\mathrm{id}\otimes[\ \ ]_{F'})\circ(\Delta^{F'}\otimes\mathrm{id})\,F^{-1}\,(F'^{\,2}\otimes 1)\ \mathrm{e}^{\,2R}(\pi^*v\otimes\pi^*u\otimes\pi^*z)\big)\\ &= s^*_{\bar p}\big([\ \ ]_{F'}\circ(\mathrm{id}\otimes[\ \ ]_{F'})\circ\,\mathrm{e}^{\,-R}\,(F'^{\,2}\otimes 1)\ \mathrm{e}^{\,2R}(\pi^*v\otimes\pi^*u\otimes\pi^*z)\big)\\ &= [\ \ ]_{\star_{\bar p}}\circ(\mathrm{id}\otimes[\ \ ]_{\star_{\bar p}})\circ\,\mathrm{e}^{\,-R}\,\big(\overline{R}_{\bar p}{}^{\alpha}(v^{\bar\phi})\otimes\overline{R}_{\bar p\,\alpha}(u^{\bar\phi})\otimes z^{\bar\phi}\big)\\ &= \big[\,\overline{R}_{\bar p}{}^{\alpha}(v^{\bar\phi})\,,\,\big[\,\overline{R}_{\bar p\alpha}(u^{\bar\phi}),z^{\bar\phi}\,\big]\,\big]^{(3)}_{\bar p}\ .\end{aligned} \tag{4.75}$$

In the second line we used $\mathcal{R}^{-1} = F'^{\,2}\,F^2$, recalled (4.52) and then the antisymmetry of the R-flux components R^{ijk}. In the third line we used Corollary 4.18, while in the fourth line we used the second identity of Proposition 4.15 and the equality $(\Delta\otimes\mathrm{id})F^{-1}\,(F'^{\,2}\otimes 1) = \mathrm{e}^{\,-2R}\,(F'^{\,2}\otimes 1)(\Delta\otimes\mathrm{id})F^{-1}$ that follows for example from the Baker-Campbell-Haudorff formula (4.10). In the fifth line we used again antisymmetry of the R-flux components and recalled that $R_{\bar p}^{-1} = F_{\bar p}^2$. We therefore conclude that the deformed Jacobi identity reads as

$$\big[u\,,\,[v,z]\big]^{(3)}_{\bar p} + \big[\,\overline{R}{}^{\beta}_{\bar p}\,\overline{R}{}^{\alpha}_{\bar p}(z)\,,\,\big[\,\overline{R}_{\bar p\,\beta}(u),\overline{R}_{\bar p\,\alpha}(v)\,\big]\,\big]^{(3)}_{\bar p} = \big[\,\overline{R}_{\bar p}{}^{\alpha}(v^{\bar\phi})\,,\,\big[\,\overline{R}_{\bar p\,\alpha}(u^{\bar\phi}),z^{\bar\phi}\,\big]\,\big]^{(3)}_{\bar p}\ . \tag{4.76}$$

These deformed Lie algebra expressions simplify considerably on the leaf with momentum $\bar p = 0$, which gives the deformed Lie algebra $\mathsf{Vect}(M)_0$ of infinitesimal diffeomeorphisms on configuration space; it is characterized by an undeformed 2-bracket $[u,v]_0^{(2)} = [u,v]$, and the 3-bracket

$$\big[u\,,\,[v,z]\big]^{(3)}_0 = [\quad]\circ(\mathrm{id}\otimes[\quad])\circ\mathrm{e}^{\,-R}(u\otimes v\otimes z) \tag{4.77}$$

that satisfies the deformed Jacobi identity

$$\big[u\,,\,[v,z]\big]^{(3)}_0 + \big[z\,,\,[u,v]\big]^{(3)}_0 + \big[v^{\bar\phi}\,,\,[z^{\bar\phi},u^{\bar\phi}]\big]^{(3)}_0 = 0\ . \tag{4.78}$$

We have derived this deformed Lie algebra structure from the one on phase space, however it is easy to verify (4.78) directly. We can also characterize the deformation via the Jacobiator

$$[u,v,z]_0 := \big[u\,,\,[v,z]\big]^{(3)}_0 + \big[z\,,\,[u,v]\big]^{(3)}_0 + \big[v\,,\,[z,u]\big]^{(3)}_0 = \big[v\,,\,[z,u]\big]^{(3)}_0 - \big[v^{\bar\phi}\,,\,[z^{\bar\phi},u^{\bar\phi}]\big]^{(3)}_0\ , \tag{4.79}$$

i.e., $[\quad]_0 = [\quad]\circ(\mathrm{id}\otimes[\quad])\circ\sigma_{12}\circ(\,\mathrm{e}^{\,-R}-1)$. We have thus derived a general expression to all orders in the R-flux components R^{ijk} for the Jacobiator on arbitrary vector fields; this formula nicely illustrates how nonassociativity can be explicitly manifest in a theory of gravity on configuration space.

Finally we study Lie derivatives of forms on configuration space. We define the Lie derivative $\mathcal{L}^{\bar p}$ by

$$\mathcal{L}^{\bar p}{}_u(\eta) := s_{\bar p}^*\circ\mathcal{L}^{\star_{\mathcal F}}_{\pi^*u}\big(\pi^*\eta\big)\ , \tag{4.80}$$

for all $u\in\mathsf{Vect}(M)_{\bar p}$, $\eta\in\Omega^\bullet(M)_{\bar p}$. As in (4.67), since no partial derivatives in the momentum directions appear, and since $s_{\bar p}^*\,\mathrm{d}p_i = 0$, we find explicitly

$$\mathcal{L}^{\bar p}{}_u(\eta) = \mathcal{L}_{\overline{F}{}^{\alpha}_{\bar p}(u)}\big(\,\overline{F}_{\bar p\,\alpha}(\eta)\,\big) =: \mathcal{L}_u^{\star_{\bar p}}(\eta) \tag{4.81}$$

where we used the notation introduced in (4.69), and $\mathcal{L}^{\star_{\bar p}}$ is the $\star_{\bar p}$ Lie derivative on Moyal–Weyl noncommutative space (cf. [4, 5]). In order to write the Leibniz rule for this infinitesimal diffeomorphism we observe from (4.43) that the exterior product $\wedge^{(2)}_{\bar p}$ is the usual exterior product in Moyal–Weyl noncommutative space: $\wedge^{(2)}_{\bar p} = \wedge_{\star_{\bar p}}$; since the Lie derivative is also the same, we have

$$\mathcal{L}^{\bar p}{}_u(\eta\wedge_{\star_{\bar p}}\omega) = \mathcal{L}^{\bar p}{}_u(\eta)\wedge_{\star_{\bar p}}\omega + \overline{R}_{\bar p}{}^{\alpha}(\eta)\wedge_{\star_{\bar p}}\mathcal{L}^{\bar p}{}_{\overline{R}_{\bar p\,\alpha}(u)}(\omega)\ . \tag{4.82}$$

In particular if we pullback this expression to the leaf $\bar p = 0$ we obtain the usual undeformed Lie derivative action.

Acknowledgments

We thank R. Blumenhagen, L. Castellani, B. Jurčo, D. Lüst, D. Minic, D. Mylonas, A. Schenkel and P. Schupp for helpful discussions, and G. Amelino-Camelia, P. Martinetti and J.-C. Wallet for the invitation to submit this contribution to their proceedings. This work was initiated while R.J.S. was visiting the Department of Theoretical Physics of the University of Torino during January–February 2014, and completed while he was visiting the Alfred Renyi Institute of Mathematics in Budapest during March–April 2015; he warmly thanks both institutions for support and hospitality during his stays there. The work of R.J.S. was supported in part by the Consolidated Grant ST/L000334/1 from the UK Science and Technology Facilities Council.

References

[1] G. Aldazabal, D. Marqués and C. Núñez, "Double field theory: A pedagogical review," Class. Quant. Grav. **30** (2013) 163001 [arXiv:1305.1907 [hep-th]].

[2] D. Andriot, M. Larfors, D. Lüst and P. Patalong, "(Non)commutative closed string on T-dual toroidal backgrounds," JHEP **1306** (2013) 021 [arXiv:1211.6437 [hep-th]].

[3] P. Aschieri, I. Bakovič, B. Jurčo and P. Schupp, "Noncommutative gerbes and deformation quantization," J. Geom. Phys. **60** (2010) 1754–1761 [arXiv:hep-th/0206101].

[4] P. Aschieri, M. Dimitrijevic, F. Meyer and J. Wess, "Noncommutative geometry and gravity," Class. Quant. Grav. **23** (2006) 1883–1912 [arXiv:hep-th/0510059].

[5] P. Aschieri, M. Dimitrijevic, P. Kulish, F. Lizzi and J. Wess, "Noncommutative spacetimes: Symmetries in noncommutative geometry and field theory," Lect. Notes Phys. **774** (2009) 1–197.

[6] I. Bakas and D. Lüst, "3-cocycles, nonassociative star products and the magnetic paradigm of R-flux string vacua," JHEP **1401** (2014) 171 [arXiv:1309.3172 [hep-th]].

[7] G.E. Barnes, A. Schenkel and R.J. Szabo, "Nonassociative geometry in quasi-Hopf representation categories I: Bimodules and their internal homomorphisms," J. Geom. Phys. **89** (2015) 111–152 [arXiv:1409.6331 [math.QA]].

[8] E. Bergshoeff, D.S. Berman, J.P. van der Schaar and P. Sundell, "A noncommutative M-theory five-brane," Nucl. Phys. B **590** (2000) 173–197 [arXiv:hep-th/0005026].

[9] D.S. Berman, M. Cederwall and M.J. Perry, "Global aspects of double geometry," JHEP **1409** (2014) 066 [arXiv:1401.1311 [hep-th]].

[10] R. Blumenhagen and E. Plauschinn, "Nonassociative gravity in string theory?," J. Phys. A **44** (2011) 015401 [arXiv:1010.1263 [hep-th]].

[11] R. Blumenhagen, M. Fuchs, F. Hassler, D. Lüst and R. Sun, "Nonassociative deformations of geometry in double field theory," JHEP **1404** (2014) 141 [arXiv:1312.0719 [hep-th]].

[12] R. Blumenhagen, A. Deser, D. Lüst, E. Plauschinn and F. Rennecke, "Non-geometric fluxes, asymmetric strings and nonassociative geometry," J. Phys. A **44** (2011) 385401 [arXiv:1106.0316 [hep-th]].

[13] C.-S. Chu and P.-M. Ho, "Noncommutative open string and D-brane," Nucl. Phys. B **550** (1999) 151–168 [arXiv:hep-th/9812219].

[14] C. Condeescu, I. Florakis and D. Lüst, "Asymmetric orbifolds, non-geometric fluxes and noncommutativity in closed string theory," JHEP **1204** (2012) 121 [arXiv:1202.6366 [hep-th]].

[15] L. Cornalba and R. Schiappa, "Nonassociative star product deformations for D-brane world-volumes in curved backgrounds," Commun. Math. Phys. **225** (2002) 33–66 [arXiv:hep-th/0101219].

[16] V.G. Drinfeld, "Quasi-Hopf algebras," Leningrad Math. J. **1** (1990) 1419–1457.

[17] L. Freidel, R.G. Leigh and D. Minic, "Born reciprocity in string theory and the nature of spacetime," Phys. Lett. B **730** (2014) 302–306 [arXiv:1307.7080 [hep-th]].

[18] P.-M. Ho, "Making nonassociative algebra associative," JHEP **0111** (2001) 026 [arXiv:hep-th/0103024].

[19] P.-M. Ho and Y.-T. Yeh, "Noncommutative D-brane in nonconstant NS–NS B-field background," Phys. Rev. Lett. **85** (2000) 5523–5526 [arXiv:hep-th/0005159].

[20] O. Hohm, D. Lüst and B. Zwiebach, "The spacetime of double field theory: Review, remarks, and outlook," Fortsch. Phys. **61** (2013) 926–966 [arXiv:1309.2977 [hep-th]].

[21] C.M. Hull, "A geometry for non-geometric string backgrounds," JHEP **0510** (2005) 065 [arXiv:hep-th/0406102].

[22] C.M. Hull, "Finite gauge transformations and geometry in double field theory," arXiv:1406.7794 [hep-th].

[23] S. Kawamoto and N. Sasakura, "Open membranes in a constant C-field background and noncommutative boundary strings," JHEP **0007** (2000) 014 [arXiv:hep-th/0005123].

[24] D. Lüst, "T-duality and closed string noncommutative (doubled) geometry," JHEP **1012** (2010) 084 [arXiv:1010.1361 [hep-th]].

[25] D. Mylonas and R.J. Szabo, "Nonassociative field theory on non-geometric spaces," Fortsch. Phys. **62** (2014) 727–732 [arXiv:1404.7304 [hep-th]].

[26] D. Mylonas, P. Schupp and R.J. Szabo, "Membrane sigma-models and quantization of non-geometric flux backgrounds," JHEP **1209** (2012) 012 [arXiv:1207.0926 [hep-th]].

[27] D. Mylonas, P. Schupp and R.J. Szabo, "Non-geometric fluxes, quasi-Hopf twist deformations and nonassociative quantum mechanics," J. Math. Phys. **55** (2014) 122301 [arXiv:1312.1621 [hep-th]].

[28] D. Mylonas, P. Schupp and R.J. Szabo, "Nonassociative geometry and twist deformations in non-geometric string theory," PoS ICMP **2013** (2013) 007 [arXiv:1402.7306 [hep-th]].

[29] C. Saemann and R.J. Szabo, "Groupoid quantization of loop spaces," PoS CORFU **2011** (2011) 046 [arXiv:1203.5921 [hep-th]].

[30] C. Saemann and R.J. Szabo, "Groupoids, loop spaces and quantization of 2-plectic manifolds," Rev. Math. Phys. **25** (2013) 1330005 [arXiv:1211.0395 [hep-th]].

[31] V. Schomerus, "D-branes and deformation quantization," JHEP **9906** (1999) 030 [arXiv:hep-th/9903205].

[32] N. Seiberg and E. Witten, "String theory and noncommutative geometry," JHEP **9909** (1999) 032 [arXiv:hep-th/9908142].

[33] J. Shelton, W. Taylor and B. Wecht, "Non-geometric flux compactifications," JHEP **0510** (2005) 085 [arXiv:hep-th/0508133].

[34] I. Vaisman, "On the geometry of double field theory," J. Math. Phys. **53** (2012) 033509 [arXiv:1203.0836 [math.DG]].